ANTICYCLOTOMIC EULER SYSTEM OVER BIQUADRATIC FIELDS

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Abstract. We construct a new Euler system (anticyclotomic, in the sense of Jetchev–Nekovář–Skinner) for the Galois representation $V_{f,\chi}$ attached to a newform f of weight $k \geq 2$ twisted by an anticyclotomic Hecke character χ defined over an imaginary biquadratic field K_0. We then show some arithmetic applications of the constructed Euler system, including results on the Bloch–Kato conjecture and a divisibility towards the Iwasawa–Greenberg main conjecture for $V_{f,\chi}$.

Contents

Date: August 4, 2025.
2020 *Mathematics Subject Classification.* Primary 11G05; Secondary 11G40.

1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N_f))$ be an elliptic newform of even weight $k = 2r \geq 2$, and let $p \nmid 6N_f$ be a prime. Let $K_0/\mathbf{Q}$ be an imaginary biquadratic field in which p splits. This means that K_0 contains two distinct imaginary quadratic subfields K_1, K_2 together with one real quadratic subfield K_3. Let L be a number field containing K_0 and the Fourier coefficients of f, and let $\mathfrak{P}$ be a prime of L above p at which f is ordinary, i.e. $v_{\mathfrak{P}}(a_p) = 0$. Let χ be an anticyclotomic Hecke character of K_0 with infinity type $(-a, a, -b, b)$ where $a \geq b \geq 0$ [1] that satisfies the decomposition hypothesis (6.1) i.e. χ can be factored

$$\chi = \tilde{\psi}_1 \tilde{\psi}_2 \mathbf{N}^{(k_1+k_2-2)/2}.$$

Here, for $i \in \{1, 2\}$, ψ_i is a Hecke character of K_i of infinity type $(1 - k_i, 0)$ and modulus $\mathfrak{f}_i$; $\tilde{\psi}_i$ is the Hecke character of K_0, obtained by composing $\mathbb{A}_{K_0}^{\times} \xrightarrow{\mathbb{N}_{K_0/K_i}} \mathbb{A}_{K_i}^{\times} \xrightarrow{\psi_i} \mathbb{C}$. Not that if this happens, we must have $k_1 = a - b + 1$ and $k_2 = a + b + 1$. We then focus on the conjugate self-dual $G_{K_0} = \mathrm{Gal}(\overline{\mathbf{Q}}/K_0)$-representation

$$V_{f,\chi} := V_f^{\vee}(1 - r) \otimes \chi^{-1},$$

where $V_f^{\vee}$ is the contragredient of Deligne's $\mathfrak{P}$-adic Galois representation associated to f.

Throughout the remainder of this section, we assume the following hypotheses:

- f is ordinary and non-Eisenstein at $\mathfrak{P}$;
- p splits completely in K_0;
- $p \nmid h_{K_0}$, where h_{K_0} is the class number of K_0.

For every integral ideal μ_3 of $\mathcal{O}_{K_3}$, let $K_0[\mu_3]$ be the maximal p-subextension of the ring class field of K_0 of conductor μ_3. Denote by $\mathcal{N}$ the set of squarefree products of primes $\mu_3 \subset \mathcal{O}_{K_3}$, where $m = N_{K_3/\mathbf{Q}}(\mu_3)$ is squarefree, prime to p, and split in K_0.

Theorem A (Theorem 3.2.1)**.** *There exists a collection of Iwasawa cohomology classes*

$$\mathbf{z}_{f,\chi,\mu_3} \in H^1_{\mathrm{Iw}}\big(K_0[\mu_3 p^{\infty}], T_{f,\chi}\big),$$

indexed by the ideals $\mu_3 \in \mathcal{N}$ with $m = N_{K_3/\mathbf{Q}}(\mu_3)$, where $T_{f,\chi}$ is a certain G_K-stable $\mathcal{O}$-lattice inside $V_{f,\chi}$, such that for every prime $\lambda_3 \in \mathcal{N}$ of norm ℓ, with $(\ell, mp) = 1$ we have the norm relation

$$\mathrm{Norm}_{K_0[\mu_3]}^{K_0[\mu_3\,\lambda_3]}(\mathbf{z}_{f,\chi,\mu_3\,\lambda_3}) = P_{\mathcal{L}_4}(\mathrm{Frob}_{\mathcal{L}_4})(\mathbf{z}_{f,\chi,\mu_3}),$$

where $P_{\mathcal{L}_4}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathcal{L}_4} \mid (T_{f,\chi})^{\vee}(1))$, and $\mathrm{Frob}_{\mathcal{L}_4}$ is the geometric Frobenius.

Remark. In [JNS], Jetchev–Nekovář–Skinner have developed a theory of 'split' anticyclotomic Euler systems attached to conjugate self-dual representations over CM fields, where classes are defined over ring class extensions of CM fields (indexed by ideals of their totally real subfields). Our construction fits within their framework. Furthermore, we note that the condition where $m = N_{K_3/\mathbf{Q}}(\mu_3)$ splits in K_0 does exclude the setting when m is inert in K_3 and μ_3 splits in K_0. Nevertheless, this does not affect the application of the [JNS] machinery (see some details for the imaginary quadratic case in [Do22, §4.3]).

Due to its geometric origin, if we let

$$\kappa_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(\mathbf{z}_{f,\chi,(1)})$$

then it will land in a Selmer subgroup of $H^1(K_0, V_{f,\chi})$ with 'nice' local conditions (see Section 6.3). Then feeding Theorem A to the general Euler system machinery of [JNS], we deduce the following cases of the Bloch–Kato conjecture in analytic rank 0.

Theorem B (Theorem 6.5.1)**.** *Let $f \in S_k(\Gamma_0(N_f))$ be a newform. Let χ be an anticyclotomic Hecke character of K_0 of infinity type $(-a, a, -b, b)$ satisfying the Hypotheses (6.1). Assume further that:*

(1) *Either $k \geq 2a + 2$ or $2b \geq k$;*

[1] By either using $L(f/K_0, \chi, r) = L(f/K_0, \chi^{\mathbf{c}}, r)$, where $\chi^{\mathbf{c}}$ is the composition of χ with the action of complex conjugation, or swapping the order of K_1 and K_2, we would be able to cover other cases of a and b.

(2) $N_f\mathcal{O}_{K_3} = \mathfrak{n}^+\mathfrak{n}^-$ *where* $\mathfrak{n}^+$ *(respectively* $\mathfrak{n}^-$*) is divisible only by primes which are split (respectively inert) in* K_0/K_3 *and* $\mathfrak{n}^-$ *is a squarefree product of an even number of primes;*
(3) $\bar{\rho}_f$ *is absolutely irreducible;*
(4) $(pN_f, \mathrm{Norm}_{K_1/\mathbf{Q}}(\mathfrak{f}_1)\mathrm{Norm}_{K_2/\mathbf{Q}}(\mathfrak{f}_2)D_{K_0}) = 1$*;*

Then

$$L(f/K_0, \chi, r) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) = 0,$$

and hence the Bloch–Kato conjecture for $V_{f,\chi}$ *holds in this case.*

Note that the first 2 conditions of Theorem B imply that the sign of the functional equation of $V_{f,\chi}$ is equal to $+1$, see also Remark 6.5.2. This puts us in an ideal situation for the non-vanishing of central $L-$values generically.

Let $\mathcal{O}$ be the ring of integers of $L_{\mathfrak{P}}$. We say that f has big image if for a certain Galois stable $\mathcal{O}$-lattice $T_f^\vee \subset V_f^\vee$, the image of $G_{\mathbf{Q}}$ in $\mathrm{Aut}_{\mathcal{O}}(T_f^\vee)$ contains a conjugate of $\mathrm{SL}_2(\mathbf{Z}_p)$. Under this assumption, we also have results towards the Bloch–Kato conjecture in the analytic rank 1 case.

Theorem C (Theorem 6.7.1)**.** *Let the hypotheses be as in Theorem B, and assume in addition that:*

(1) $\bar{\rho}_f$ *is* p*-distinguished;*
(2) f *has big image;*
(3) $p > k - 2$.

If $2a \geq k \geq 2b + 2$ *(which implies* $L(f/K, \chi, r) = 0$*), then*

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) \geq 1.$$

Finally, we note that these results also include the proof of a divisibility towards the anticyclotomic Iwasawa Main Conjecture for $V_{f,\chi}$, see Theorem 6.6.1.

1.1. **Relation to previous works.** When χ is an anticyclotomic Hecke character over K, an imaginary quadratic field, the arithmetic of $V_{f,\chi}$ has been studied intensively via the Euler system of Heegner points pioneered by Gross–Zagier and Kolyvagin [GZ86, Kol88] (see also [Zha97, Tia03, Nek07]), and generalized Heegener cycles by Bertolini–Darmon–Prasanna [BDP13]. In particular, these objects have direct implications towards the Bloch-Kato conjecture in analytic rank 0 for $V_{f,\chi}$ by either varying the generalised Heegner cycles in p-adic families like in Castella–Hsieh [CH18] (see also [Cas20]), or by the 'level-raising' method like in Bertolini–Darmon [BD05] (see also [LV10, CH15, Chi17]). In the same vein as [BD05], Nekovář [Nek12] and Wang [Wan23] proved results towards the rank 0 Bloch-Kato conjecture when f is a cuspidal Hilbert modular eigenform over a totally real field F of parallel weight 2 and higher weights respectively, where χ is a finite order character, see also result of Tamiozzo [Tam21].

Outside of the Heegner realm, it is worthwhile to mention that the Euler system of Beilinson–Flach classes constructed by Lei–Loeffler–Zerbes [LLZ14, LLZ15] and Kings–Loeffler–Zerbes [KLZ17, KLZ20] can be applied to obtain similar rank 0 results. Relying on this, Lamplugh [Lam] constructed Euler systems for $\mathrm{Ind}_{K_0}^{K_1}\mathcal{O}(\chi\rho)$ over K_1 (where ρ is an auxiliary character) and used that to bound the associated Selmer group over the K_0 via Rubin's machinery [Rub00].

The anticyclotomic Euler system over K_0 that we will describe in this paper is more comparable with the anticyclotomic diagonal Euler system [Do22, CD23] over K (an imaginary quadratic field) and comes together with application towards the Bloch-Kato conjecture in analytic rank 0. The construction of the cohomology classes, similar to [Do22, CD23], is based on a generalisation of the diagonal cycles pioneered by Gross–Kudla [GK92] and Gross–Schoen [GS95], and improved recently by Darmon–Rotger and Bertolini–Seveso–Venerucci (see [BDR+22]). Despite the fact that it is being done later, the imaginary biquadratic case is actually a generic case (where $K_1 \neq K_2$) while the imaginary quadratic case is a degenerate situation (where $K = K_1 = K_2$).

In future work, we intend to construct a bipartite Euler system over a biquadratic field as well as investigate the case where p does not split completely in K_0.

1.2. **Acknowledgements.** The author would like to thank Francesc Castella and Christopher Skinner for many fruitful discussions and encouragements, Haruzo Hida for suggesting the problem, Chandrashekhar Khare, Romyar Sharifi and Alex Smith for helpful communication, and the anonymous referee for many helpful comments and suggestions on an earlier draft of this paper.

2. Preliminaries

2.1. **Galois representations associated to newforms.** In this section, we follow [CD23, Sec. 1.1] and introduce some important notation and results. Let $f \in S_k(\Gamma_1(N_f), \chi_f)$ be a normalized newform of weight $k \geq 2$ and let $\sum_{n=1}^{\infty} a_n q^n$ be its q-expansion. Let $p \nmid N_f$ be a prime. Fix embeddings $i_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $L/\mathbf{Q}$ be the coefficient field of f that is, L contains all values $i_\infty^{-1}(a_n)$ and $i_\infty^{-1} \circ \chi_f$. Let $\mathfrak{P}$ be the prime of L above p with respect to i_p. Let $S = \{\text{prime } \ell | pN_f\} \cup \{\infty\}$. Then Eichler-Shimura (for $k = 2$) and Deligne (for $k > 2$) construct a $p-$adic Galois representation associated to f:

$$\rho_{f,\mathfrak{P}} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(L_{\mathfrak{P}}),$$

such that for all primes $\ell \notin S$:

- $\mathrm{trace}(\rho_{f,\mathfrak{P}}(\mathrm{Frob}_\ell)) = i_p(a_\ell)$,
- $\det(\rho_{f,\mathfrak{P}}(\mathrm{Frob}_\ell)) = i_p(\chi_f(\ell)\ell^{k-1})$,
- $\rho_{f,\mathfrak{P}}$ is irreducible, hence absolutely irreducible.

Here Frob_ℓ is the geometric Frobenius.

As in [CD23, Sec. 1.1], one obtains the geometric realization V_f of $\rho_{f,\mathfrak{P}}$ defined as the subspace of

$$H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{S}_{k-2}) \otimes L_{\mathfrak{P}}.$$

Dually, $V_f^\vee = \mathrm{Hom}(V_f, L_{\mathfrak{P}})$ can be interpreted as the maximal quotient of

$$H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \otimes L_{\mathfrak{P}}$$

on which the dual Hecke operator T'_ℓ acts as multiplication by a_ℓ for all $\ell \nmid N_f p$ and $\langle d \rangle = \langle d \rangle^*$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbf{Z}/N_f\mathbf{Z})^\times$.

Let $\mathcal{O}$ be the ring of integers of $L_{\mathfrak{P}}$. There exists a $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice $T_f^\vee \subset V_f^\vee$ defined as the image of $H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \otimes \mathcal{O}$ in $V_f^\vee$.

If f is ordinary at p (which means $i_p(a_p) \in \mathcal{O}^\times$), then the restriction of V_f to $G_{\mathbf{Q}_p}$ is reducible. This leads us to an exact sequence of $L_{\mathfrak{P}}[G_{\mathbf{Q}_p}]$-modules

$$0 \to V_f^+ \to V_f \to V_f^- \to 0,$$

where $\dim_{L_{\mathfrak{P}}} V_f^\pm = 1$. Dually, we also obtain an exact sequence for the restriction of $V_f^\vee$ to $G_{\mathbf{Q}_p}$

$$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0, \tag{2.1}$$

where $V_f^{\vee,+} \simeq (V_f^-)^\vee(1-k)(\chi_f^{-1})$, and the $G_{\mathbf{Q}_p}$-action on the quotient $V_f^{\vee,-}$ is given by the unramified character sending the arithmetic Frobenius Frob_p^{-1} to α_p, which is the unit root of $x^2 - a_p x + \chi_f(p)p^{k-1}$.

2.2. **Patched CM Hecke modules.** Here, we recall the conventions on Hecke characters and the construction of certain patched CM Hecke modules from [CD23, Sec. 1.3] and [LLZ15].

2.2.1. *Hecke characters and theta series.* Let K be an imaginary quadratic field. Let $p = \mathfrak{p}\bar{\mathfrak{p}}$ be a prime that splits in K with $\mathfrak{p}$, the prime of K above p, induced by $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We say that a Hecke character $\psi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ has infinity type (m, n), where m, n are integers, if $\psi_\infty(x_\infty) = x_\infty^m \bar{x}_\infty^n$.

Let $\mathrm{rec}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$ be the geometrically normalized Artin reciprocity map. Following [CD23, Sec. 1.3.1], given $g \in G_K$, we take $x \in \mathbb{A}_K^\times$ such that $\mathrm{rec}_K(x) = g|_{K^{\mathrm{ab}}}$ and define

$$\psi_{\mathfrak{P}}(g) = i_p \circ i_\infty^{-1}(\psi(x) x_\infty^{-m} \bar{x}_\infty^{-n}) x_{\mathfrak{p}}^m x_{\bar{\mathfrak{p}}}^n.$$

Such a $\psi_{\mathfrak{P}}$ will be called the p-adic avatar of ψ. We shall also use ψ to denote its p-adic avatar if the context makes this usage reasonable.

Attached to ψ, a Hecke character of K of infinity type $(-1,0)$ with conductor $\mathfrak{f}$ that takes values in a finite extension L/K, is the theta series

$$\theta_\psi = \sum_{(\mathfrak{a},\mathfrak{f})=1} \psi(\mathfrak{a}) q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in S_2(\Gamma_1(N_\psi), \chi_\psi \epsilon_K)$$

where $N_\psi = N_{K/\mathbf{Q}}(\mathfrak{f})\mathrm{disc}(K/\mathbf{Q})$, χ_ψ is the unique Dirichlet character modulo $N_{K/\mathbf{Q}}(\mathfrak{f})$ such that $\psi((n)) = n\chi_\psi(n)$ for all $n \in \mathbf{Z}$ with $(n, N_{K/\mathbf{Q}}(\mathfrak{f})) = 1$, and ϵ_K is the quadratic Dirichlet character attached to K. The cuspform θ_ψ is new of level $N_\psi = N_{K/\mathbf{Q}}(\mathfrak{f}) \cdot \mathrm{disc}(K/\mathbf{Q})$ by [Miy89]. One obtains the following description of the $\mathfrak{P}$-adic representation of θ_ψ

$$V_{\theta_\psi}^\vee \cong \mathrm{Ind}_K^{\mathbf{Q}} L_{\mathfrak{P}}(\psi^{-1}),$$

where $\mathfrak{P}$ is the prime of L above p with respect to i_p.

2.2.2. *Hecke algebras and norm maps.* We keep the notation of the previous section and follow [CD23, Sec. 1.3.1]. Let $\mathfrak{n} \subset \mathcal{O}_K$ be an ideal divisible by $\mathfrak{f}$ and let $N = N_{K/\mathbf{Q}}(\mathfrak{n})\mathrm{disc}(K/\mathbf{Q})$. Let $K_\mathfrak{n}$ be the ray class field of K with conductor $\mathfrak{n}$. Let $H_\mathfrak{n} = \mathrm{Gal}(K_\mathfrak{n}/K)$ be the ray class group of K modulo $\mathfrak{n}$. Let $K(\mathfrak{n})$ be the largest p-subextension of K contained in $K_\mathfrak{n}$, i.e. $\mathrm{Gal}(K(\mathfrak{n})/K) \cong H_\mathfrak{n}^{(p)}$ is the largest p-power quotient of $H_\mathfrak{n}$. Given an ideal $\mathfrak{k}$ of K that is coprime to $\mathfrak{n}$, let $[\mathfrak{k}]$ be the class of $\mathfrak{k}$ in $H_\mathfrak{n}$. Let $\mathbb{T}'(N)$ be the subalgebra of $\mathrm{End}_\mathbf{Z}(H^1(Y_1(N)(\mathbf{C}), \mathbf{Z}))$ generated by $\langle d\rangle'$ and T'_ℓ for all primes ℓ, then one can prove that:

Proposition 2.2.1 (Proposition 3.2.1 in [LLZ15])**.** *There exists a homomorphism $\phi_\mathfrak{n} : \mathbb{T}'(N) \to \mathcal{O}[H_\mathfrak{n}]$ defined by*

$$\phi_\mathfrak{n}(T'_\ell) = \sum_{\substack{\mathfrak{l}\subset\mathcal{O}_K, \mathfrak{l}\nmid\mathfrak{n},\\ N_{K/\mathbf{Q}}(\mathfrak{l})=\ell}} [\mathfrak{l}]\psi(\mathfrak{l}),$$

$$\phi_\mathfrak{n}(\langle d\rangle') = \chi_\psi(d)\epsilon_K(d)[(d)].$$

For $\mathfrak{m} = \mathfrak{n}\mathfrak{l}$, with $\mathfrak{l}$ a prime ideal and $(\mathfrak{m}, p) = 1$, put $M = N_{K/\mathbf{Q}}(\mathfrak{m})\mathrm{disc}(K/\mathbf{Q})$ and one has the following map

$$\mathcal{N}_\mathfrak{n}^\mathfrak{m} : \mathcal{O}[H_\mathfrak{m}^{(p)}] \otimes_{\mathbb{T}'(M)\otimes\mathbf{Z}_p,\phi_\mathfrak{m}} H^1_{\text{ét}}(Y_1(M)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \to \mathcal{O}[H_\mathfrak{n}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbf{Z}_p,\phi_\mathfrak{n}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)).$$

This norm map is defined explicitly by splitting into 3 cases (see [CD23, Sec. 1.1.2] for the definition of the degeneracy map):

- If $\mathfrak{l} \mid \mathfrak{n}$ then
$$\mathcal{N}_\mathfrak{n}^\mathfrak{m} = 1 \otimes \mathrm{pr}_{1*};$$
- If $\mathfrak{l} \nmid \mathfrak{n}$ is split or ramified in K and $N_{K/\mathbf{Q}}(\mathfrak{l}) = \ell$, then
$$\mathcal{N}_\mathfrak{n}^\mathfrak{m} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell} \otimes \mathrm{pr}_{\ell*};$$
- If $\mathfrak{l} \nmid \mathfrak{n}$ is inert in K, say $\mathfrak{l} = (\ell)$, then
$$\mathcal{N}_\mathfrak{n}^\mathfrak{m} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell^2} \otimes \mathrm{pr}_{\ell\ell*}.$$

Note that one can extend the definition of $\mathcal{N}_\mathfrak{n}^\mathfrak{m}$ to any pair of ideals $\mathfrak{n} \mid \mathfrak{m}$ by composition.

Following [CD23, Sec. 1.3.2], if p splits in K and $(p, \mathfrak{f}) = 1$ then for any ideal $\mathfrak{n} \subset \mathcal{O}_K$ divisible by $\mathfrak{f}$ such that $(\mathfrak{n}, \bar{\mathfrak{p}}) = 1$, the maximal ideal of $\mathbb{T}'(N)$ defined by the kernel of the composition

$$\mathbb{T}'(N) \xrightarrow{\phi_\mathfrak{n}} \mathcal{O}[H_\mathfrak{n}] \xrightarrow{\text{aug}} \mathcal{O} \to \mathcal{O}/\mathfrak{P},$$

is non-Eisenstein, p-ordinary, and p-distinguished.

We finish this section by extracting a crucial result in [LLZ15] in the case where p splits in K. This will be used later to prove the norm relation of our Euler system.

Theorem 2.2.2 (Corollary 5.2.6 in [LLZ15])**.** *Assume that $(p, \mathfrak{f}) = 1$. Let $\mathcal{A}$ be the set of ideals $\mathfrak{m} \subset \mathcal{O}_K$ with $(\mathfrak{m}, \bar{\mathfrak{p}}) = 1$, and put $\mathcal{A}_{\mathfrak{f}} = \{\mathfrak{f}\mathfrak{m} \colon \mathfrak{m} \in \mathcal{A}\}$. Given $\mathfrak{n} \in \mathcal{A}_{\mathfrak{f}}$, there is a $G_{\mathbf{Q}}$-equivariant isomorphism of $\mathcal{O}[H_{\mathfrak{n}}^{(p)}]$-modules*

$$\nu_{\mathfrak{n}} : \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes \mathbf{Z}_p, \phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \xrightarrow{\cong} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{n})} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}).$$

Furthermore, for any $\mathfrak{n}, \mathfrak{m} \in \mathcal{A}_{\mathfrak{f}}$ with $\mathfrak{n} \mid \mathfrak{m}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathbb{T}'(M)\otimes \mathbf{Z}_p, \phi_{\mathfrak{m}}} H^1_{\text{ét}}(Y_1(M)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \arrow[r, "\nu_{\mathfrak{m}}", "\cong"'] \arrow[d, "\mathcal{N}_{\mathfrak{n}}^{\mathfrak{m}}"'] & \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{m})} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \arrow[d, "\mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{m}}"'] \\
\mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes \mathbf{Z}_p, \phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \arrow[r, "\nu_{\mathfrak{n}}", "\cong"'] & \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{n})} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}),
\end{tikzcd}$$

where $\mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{m}}$ is the natural norm map of the induced representations.

3. The construction

For a newform f and two Hecke characters ψ_1, ψ_2 of 2 distinct imaginary quadratic fields K_1, K_2 respectively, using the results from [CD23], [BSV22] and [LLZ15] recalled in the preceding section, we construct a family of cohomology classes for $f \otimes \tilde{\psi}_1\tilde{\psi}_2$ defined over ring class field extensions of K_0, which is the compositum of K_1 and K_2, and prove that they satisfy the norm relations of an anticyclotomic Euler system. Following [CD23, Sec. 2], we first give the construction and show the tame norm relations in the case where $(f, \theta_{\psi_1}, \theta_{\psi_2})$ have weights $(2, 2, 2)$. Then by varying the diagonal cycle classes in Hida families we extend the construction to more general weights and prove the wild norm relations.

Throughout this section we consider the following set-up:

(1) Let $f \in S_k(\Gamma_0(N_f))$ be a newform of weight $k \geq 2$.
(2) Let $K_1/\mathbf{Q}$ be an imaginary quadratic field of discriminant D_1 coprime to N_f. Let ψ_1 be a Hecke character of K_1 of infinity type $(1 - k_1, 0)$, with $k_1 \geq 1$, and modulus $\mathfrak{f}_1$.
(3) Let $K_2/\mathbf{Q}$ be an imaginary quadratic field of discriminant $D_2 \neq D_1$ and coprime to N_f. Let ψ_2 be a Hecke character of K_2 of infinity type $(1 - k_2, 0)$, with $k_2 \geq 1$, and modulus $\mathfrak{f}_2$.
(4) Denote by ϵ_{K_i} the quadratic character attached to the quadratic field K_i for $i \in \{1, 2\}$.
(5) Let K_0 be the compositum of K_1 and K_2. Since K_0 is a biquadratic field, we can consider K_3, the unique real quadratic field inside K_0.
(6) Let $\tilde{\psi}_i$ be the Hecke character of K_0, obtained by composing $\mathbb{A}_{K_0}^\times \xrightarrow{\mathbb{N}_{K_0/K_i}} \mathbb{A}_{K_i}^\times \xrightarrow{\psi_i} \mathbb{C}$ for $i \in \{1, 2\}$.
(7) Denote by

$$\theta_{\psi_i} \in S_{k_i}(N_{\psi_i}, \chi_{\psi_i}\epsilon_{K_i})$$

the associated theta series, where $N_{\psi_i} = N_{K_i/\mathbf{Q}}(\mathfrak{f}_i) \cdot \mathrm{disc}(K_i/\mathbf{Q})$ and χ_{ψ_i} is the Dirichlet character modulo $N_{K_i/\mathbf{Q}}(\mathfrak{f}_i)$ defined by $\psi_i((n)) = n^{k_i - 1}\chi_{\psi_i}(n)$ for all integers n prime to $N_{K_i/\mathbf{Q}}(\mathfrak{f}_i)$ $(i \in \{1, 2\})$.
(8) We assume the self-duality condition

$$\chi_{\psi_1}\epsilon_{K_1}\chi_{\psi_2}\epsilon_{K_2} = 1. \tag{3.1}$$

Let L/K_0 be a finite extension containing the Fourier coefficients of f, θ_{ψ_1}, and θ_{ψ_2}. Let $p \geq 5$ be a prime that splits in K_0 and such that $(p, N_f N_{\psi_1} N_{\psi_2}) = 1$, and let $\mathfrak{P}|\mathfrak{p}$ be the prime of L/K_0 above p determined by a fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Finally, let $L_{\mathfrak{P}}$ be the completion of L at $\mathfrak{P}$, and denote by $\mathcal{O}$ the ring of integers of $L_{\mathfrak{P}}$.

3.0.1. *Digression to primes decomposition and the top left-corner notations.* Let ℓ be a split prime in K_0 i.e. $(\ell) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$. We can write $(\ell) = \lambda_1\,\bar{\lambda}_1$ and $(\ell) = \lambda_2\,\bar{\lambda}_2$ in K_1 and K_2 respectively. Note that ℓ also splits in K_3 as $\lambda_3\,\tilde{\lambda}_3$, where the tilde corresponds to the nontrivial element generating the Galois group $\mathrm{Gal}(K_3/\mathbf{Q})$.

Let τ_i to be the generator of $\mathrm{Gal}(K_0/K_i)$ for $i = \{1, 2, 3\}$ then we have $\tau_3 = \tau_1\tau_2$ (this is the complex conjugation on K_3). Due to the Galois group action on primes lying above ℓ, we can further assume that:

$$\lambda_1 = \mathcal{L}_1\mathcal{L}_4, \qquad \bar{\lambda}_1 = \mathcal{L}_3\mathcal{L}_2, \qquad \lambda_2 = \mathcal{L}_1\mathcal{L}_3, \qquad \bar{\lambda}_2 = \mathcal{L}_2\mathcal{L}_4,$$
$$\lambda_3 = \mathcal{L}_4\mathcal{L}_3 \qquad (\text{so } \lambda_3 \mid \lambda_1\,\lambda_2), \qquad \text{and} \qquad \tilde{\lambda}_3 = \mathcal{L}_1\mathcal{L}_2,$$

where

$$\mathcal{L}_4 = \tau_1\mathcal{L}_1, \qquad \mathcal{L}_3 = \tau_2\mathcal{L}_1, \qquad \mathcal{L}_2 = \tau_3\mathcal{L}_1 = \tau_1\tau_2\mathcal{L}_1.$$

Denote by $\mathcal{L}$ the set of primes $\lambda_3 \subset \mathcal{O}_{K_3}$, where $\ell = N_{K_3/\mathbf{Q}}(\lambda_3)$ is prime to p and ℓ splits in K_0. Let $\mathcal{N}$ be the set of squarefree products of primes inside $\mathcal{L}$ such that its norm down to $\mathbf{Q}$ is still squarefree. For such λ_3, we can choose $\lambda_1 \subset \mathcal{O}_{K_1}$ and $\lambda_2 \subset \mathcal{O}_{K_2}$ as above such that $\lambda_3 \mid \lambda_1\,\lambda_2$.

Let $\mu_3 \in \mathcal{N}$ and $N_{K_3/\mathbf{Q}}(\mu_3) = m$. Then its norm $m = \prod_i \ell_i$ will be a product of split primes ℓ_i in K_0. Similarly, we can decompose $(m) = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\mathcal{M}_4$, $(m) = \mu_1\bar{\mu}_1$, $(m) = \mu_2\bar{\mu}_2$, $(m) = \mu_3\tilde{\mu}_3$ as a decomposition inside K_0, K_1, K_2 and K_3 respectively, where we can have the following decomposition:

$$\mu_1 = \mathcal{M}_1\mathcal{M}_4, \qquad \bar{\mu}_1 = \mathcal{M}_3\mathcal{M}_2, \qquad \mu_2 = \mathcal{M}_1\mathcal{M}_3, \qquad \bar{\mu}_2 = \mathcal{M}_2\mathcal{M}_4,$$
$$\mu_3 = \mathcal{M}_4\mathcal{M}_3 \qquad (\text{so } \mu_3|\mu_1\mu_2), \qquad \text{and} \qquad \tilde{\mu}_3 = \mathcal{M}_1\mathcal{M}_2.$$

Here, for every i, $\mathcal{M}_j = \prod_i \mathcal{L}_{j,i}$, $\ell_i = \prod_j \mathcal{L}_{j,i}$, for $1 \le j \le 4$, $\mu_j = \prod_i \lambda_{j,i}$ for every $j \in \{1, 2, 3\}$.

For each $i \in \{0, 1, 2\}$, we denote ${}^iK_{\mathfrak{n}_i}$ as the ray class field of K_i with conductor $\mathfrak{n}_i$ (an integral ideal inside $\mathcal{O}_{K_i}$), and let ${}^iH_{\mathfrak{n}_i}$ be the ray class group of K_i modulo $\mathfrak{n}_i$. Let $K_i(\mathfrak{n}_i)$ be the largest p-subextension of K_i contained in ${}^iK_{\mathfrak{n}_i}$, so $\mathrm{Gal}(K_i(\mathfrak{n}_i)/K_i) \cong {}^iH^{(p)}_{\mathfrak{n}_i}$ is the largest p-power quotient of ${}^iH_{\mathfrak{n}_i}$.

3.1. **Construction in weight $(2, 2, 2)$ and tame norm relation.** Suppose in this subsection that $(k, k_1, k_2) = (2, 2, 2)$. Let $N = \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})$. Following Section 2.1 of [CD23], which is based on the diagonal classes in the triple product of modular curves [BSV22, Sec. 3], we have cohomology classes:

(3.2)
$$\mathcal{Z}^{(1)}_m := \tilde{\kappa}^{(3)}_m \in H^1\big(\mathbf{Q}, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))(-1)\big).$$

for every positive integer m. One then chooses a test vector $\breve{f} \in S_2(N)[f]$. As noted in op. cit., the maps used to construct $\mathcal{Z}^{(1)}_m$ are compatible with correspondences. This allows one to tensor them with $\mathcal{O}$ and obtain:

$$\mathcal{Z}^{(1)}_{\mu_3} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1\mu_1}]$$
$$\otimes H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2\mu_2}](-1)\big).$$

Here, the chosen $\breve{f}$ is used to take the image under the projection $H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \to T_f^\vee$ in the first factor. The tensor products are taken from Proposition 2.2.1

$$\phi_{\mathfrak{f}_1\mu_1} : \mathbb{T}'(N_{\psi_1}m) \to \mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1\mu_1}], \quad \phi_{\mathfrak{f}_2\mu_2} : \mathbb{T}'(N_{\psi_2}m) \to \mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2\mu_2}]$$

with respect to two distinct imaginary quadratic fields K_1 and K_2, respectively .

Via the isomorphisms from Proposition 2.2.2 with respect to 2 distinct imaginary quadratic fields:

$$\nu_{\mathfrak{f}_1\mu_1} : H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1\mu_1}] \xrightarrow{\sim} \mathrm{Ind}^{\mathbf{Q}}_{K_1(\mathfrak{f}_1\mu_1)}\mathcal{O}(\psi_1^{-1}),$$
$$\nu_{\mathfrak{f}_2\mu_2} : H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2\mu_2}] \xrightarrow{\sim} \mathrm{Ind}^{\mathbf{Q}}_{K_2(\mathfrak{f}_2\mu_2)}\mathcal{O}(\psi_2^{-1}),$$

one then obtains a cohomology class in

$$H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K_1(\mathfrak{f}_1\mu_1)}\mathcal{O}(\psi_1^{-1}) \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K_2(\mathfrak{f}_2\mu_2)}\mathcal{O}(\psi_2^{-1})(-1)\big),$$

which under the maps ${}^1H_{\mathfrak{f}_1\mu_1} \twoheadrightarrow {}^1H_{\mu_1}$ and ${}^2H_{\mathfrak{f}_2\mu_2} \twoheadrightarrow {}^2H_{\mu_2}$ can be projected to a class

$$\mathcal{Z}^{(2)}_{\mu_3} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K_1}\mathcal{O}_{\psi_1^{-1}}[{}^1H^{(p)}_{\mu_1}] \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K_2}\mathcal{O}_{\psi_2^{-1}}[{}^2H^{(p)}_{\mu_2}](-1)\big). \tag{3.3}$$

Note that the group cohomology can be rewritten as

$$H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K_0}\mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[{}^1H^{(p)}_{\mu_1} \times {}^2H^{(p)}_{\mu_2}](-1)\big),$$

which by Shapiro's lemma gives us elements:

$$\mathcal{Z}^{(3)}_{\mu_3} \in H^1\big(K_0, T_f^\vee \otimes_{\mathcal{O}} \mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[{}^1H^{(p)}_{\mu_1} \times {}^2H^{(p)}_{\mu_2}](-1)\big). \tag{3.4}$$

3.1.1. *Projection to ring class groups.* Recall the fundamental exact sequence for ray class groups:

$$\mathcal{O}^\times_{K_i} \longrightarrow (\mathcal{O}_{K_i}/\mu_i\mathcal{O}_{K_i})^\times \longrightarrow {}^iH_{\mu_i} \longrightarrow {}^iH_1 \longrightarrow 1,$$

where $i \in \{1,2\}$. Assume that $p \nmid 6h_{K_0}$, where h_{K_0} is the class number of K_0. Note that by [FT91, Thm. 74], we have

$$p \nmid h_{K_i},$$

which is the class number of K_i for $i \in \{1,2,3\}$. Taking the p-primary parts of the above exact sequence induces two isomorphisms

$${}^1H^{(p)}_{\mu_1} \xrightarrow{\simeq} (\mathcal{O}_{K_1}/\mu_1\mathcal{O}_{K_1})^{\times,(p)} \xrightarrow{\simeq} (\mathcal{O}_{K_0}/\mathcal{M}_4\mathcal{O}_{K_0})^{\times,(p)}$$
$${}^2H^{(p)}_{\mu_2} \xrightarrow{\simeq} (\mathcal{O}_{K_2}/\mu_2\mathcal{O}_{K_2})^{\times,(p)} \xrightarrow{\simeq} (\mathcal{O}_{K_0}/\mathcal{M}_3\mathcal{O}_{K_0})^{\times,(p)}$$

and hence the following projection:

$${}^1H^{(p)}_{\mu_1} \times {}^2H^{(p)}_{\mu_2} \xrightarrow{\simeq} (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^{\times,(p)} \twoheadrightarrow {}^0H^{(p)}_{\mu_3}. \tag{3.5}$$

Recall that K_3 is the totally real field sitting inside a CM field K_0. Given an integral ideal $\mathfrak{n}$ of K_3, let $H[\mathfrak{n}]$ be the ring class group of K_0 of conductor $\mathfrak{n}$, so $H[\mathfrak{n}] \simeq \mathrm{Pic}(\mathcal{O}_\mathfrak{n})$ under the Artin reciprocity map, where $\mathcal{O}_\mathfrak{n} = \mathcal{O}_{K_3} + \mathfrak{n}\mathcal{O}_{K_0}$ is the order of K_0 of conductor $\mathfrak{n}$. Let $H[\mathfrak{n}]^{(p)}$ be the maximal p-power quotient of $H[\mathfrak{n}]$, and denote by $K_0[\mathfrak{n}]$ be the maximal p-extension inside the ring class field of K_0 of conductor $\mathfrak{n}$, i.e. $H[\mathfrak{n}]^{(p)} = \mathrm{Gal}(K_0[\mathfrak{n}]/K_0)$. Note that for the ring class groups and fields of K_0, we drop the upper left corner 0 notation.

Proposition 3.1.1. *Suppose $p \nmid 6h_{K_0}$ and μ_3 is a squarefree ideal of $\mathcal{O}_{K_3}$ of norm m, where m is divisible only by primes that are split in K_0. Using (3.5), we have the following short exact sequence:*

$$1 \to (\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} \xrightarrow{\Delta} {}^1H^{(p)}_{\mu_1} \times {}^2H^{(p)}_{\mu_2} \xrightarrow{e\circ w} H[\mu_3]^{(p)} \to 1.$$

Here, the map Δ uses the identifications

$$(\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} \simeq (\mathcal{O}_{K_0}/\mathcal{M}_4\mathcal{O}_{K_0})^{\times,(p)}, \qquad (\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} \simeq (\mathcal{O}_{K_0}/\mathcal{M}_3\mathcal{O}_{K_0})^{\times,(p)}.$$

Moreover, if $(\ell) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$ is a prime that splits in K_0 and is coprime to m, the projection $e \circ w$ (defined in the proof below) sends

$$[\lambda_1] \times [\bar{\lambda}_2] \mapsto \mathrm{Frob}_{\mathcal{L}_4/\lambda_3}$$

where $\mathrm{Frob}_{\mathcal{L}_4/\lambda_3}$ is the Frobenius element of $\mathcal{L}_4$ in $H[\mu_3]^{(p)}$.

Proof. We have the following exact diagram:

$$\begin{tikzcd}
\mathcal{O}^\times_{K_0} \arrow[r] \arrow[d] & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^\times \arrow[r] \arrow[d] & {}^0H_{\mu_3} \arrow[r] \arrow[d] & {}^0H_1 \arrow[r] \arrow[d] & 1 \\
\mathcal{O}^\times_{K_0}/\mathcal{O}^\times_{K_3} \arrow[r] & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^\times/(\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^\times \arrow[r] & H[\mu_3] \arrow[r] & {}^0H_1 \arrow[r] & 1.
\end{tikzcd}$$

Taking the p−part of this, using the assumption that $p \nmid 6h_{K_0}$ together with the fact that $|\mathcal{O}_{K_0}^\times/\mathcal{O}_{K_3}^\times|$ is a power of 2, we obtain the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{K_0}^\times \otimes \frac{\mathbf{Q}_p}{\mathbf{Z}_p} \arrow[r] \arrow[d] & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^{\times,(p)} \arrow[r, "w"] \arrow[d] & {}^0H_{\mu_3}^{(p)} \arrow[r] \arrow[d, "e"] & 1 \\
1 \arrow[r] & (\mathcal{O}_{K_0}/\mu_3\mathcal{O}_{K_0})^{\times,(p)}/(\mathcal{O}_{K_3}/\mu_3\mathcal{O}_{K_3})^{\times,(p)} \arrow[r] & H[\mu_3]^{(p)} \arrow[r] & 1.
\end{tikzcd}$$

Using the middle arrow and the identification 3.5, we can show the first exact sequence.

One can show the second part by noting that $\mathcal{L}_4$ is a prime of K_0 lying above both λ_1 (a prime of K_1) and $\bar{\lambda}_2$ (a prime of K_2). □

In the same setting as Proposition 3.1.1, we can consider the image of $\mathcal{Z}_{\mu_3}^{(3)}$ from 3.4 under the composition $e \circ w$. This results in the class

$$\mathcal{Z}_{\mu_3}^{(4)} \in H^1\big(K_0, T_f^\vee \otimes_{\mathcal{O}} \mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3]^{(p)}](-1)\big).$$

By Shapiro's lemma, its image under the isomorphism

$$H^1\big(K_0, T_f^\vee \otimes_{\mathcal{O}} \mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3]^{(p)}](-1)\big) \simeq H^1\big(K_0[\mu_3], T_f^\vee(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1})(-1)\big)$$

then defines

$$\mathcal{Z}_{\mu_3}^{(5)} \in H^1\big(K_0[\mu_3], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big).$$

The next lemma is in the same vein with [CD23, Lem. 2.1.3]:

Lemma 3.1.2. *Let λ_3 be a split prime in K_3 of norm ℓ coprime to mp, where ℓ splits in K_0. The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Ind}_{K_1}^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[{}^1H_{\mu_1\lambda_1}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_{K_2}^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[{}^2H_{\mu_2\lambda_2}^{(p)}] \arrow[r] \arrow[d, "\mathrm{Norm}_{\mu_1}^{\mu_1\lambda_1}\otimes\mathrm{Norm}_{\mu_2}^{\mu_2\lambda_2}"] & \mathrm{Ind}_{K_0}^{K_3}\mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3\lambda_3]^{(p)}] \arrow[d, "\mathrm{Norm}_{\mu_3}^{\mu_3\lambda_3}"] \\
\mathrm{Ind}_{K_1}^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[{}^1H_{\mu_1}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_{K_2}^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[{}^2H_{\mu_2}^{(p)}] \arrow[r] & \mathrm{Ind}_{K_0}^{K_3}\mathcal{O}_{\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}}[H[\mu_3]^{(p)}],
\end{tikzcd}$$

where the norm maps are the natural ones, and where the horizonal arrows are given by the composition $e \circ w$ in (3.1.1).

Using this Lemma 3.1.2, one can show the following Proposition 3.1.3. Similar to [CD23, Prop 2.2.1], this is the key result for the construction of our anticyclotomic Euler system for $T_f^\vee(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1})(-1)$ over the biquadratic field K_0.

Proposition 3.1.3. *The family $\{\mathcal{Z}_{\mu_3}^{(5)}\}$ satisfies the following norm relation:*

$$\begin{aligned}\mathrm{Norm}_{K_0[\mu_3]}^{K_0[\mu_3\lambda_3]}(\mathcal{Z}_{\mu_3\lambda_3}^{(5)}) = (\ell-1)\bigg(&a_\ell(f) - \frac{\psi_1(\lambda_1)\psi_2(\bar{\lambda}_2)}{\ell}([\lambda_1]\times[\bar{\lambda}_2]) - \frac{\psi_1(\bar{\lambda}_1)\psi_2(\lambda_2)}{\ell}([\bar{\lambda}_1]\times[\lambda_2]) \\ &+ (1-\ell)\frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1]\times[\lambda_2])\bigg)(\mathcal{Z}_{\mu_3}^{(5)}).\end{aligned}$$

Proof. As in the proof of [CD23, Prop 2.2.1], one has

$$\begin{aligned}
&(1\otimes\mathcal{N}^{\mu_1\,\lambda_1}_{\mu_1}\otimes\mathcal{N}^{\mu_2\,\lambda_2}_{\mu_2})(\mathcal{Z}^{(5)}_{\mu_3\,\lambda_3})\\
&=(\ell-1)\bigg((T_\ell,1,1)-\frac{\psi_1(\lambda_1)[\lambda_1]}{\ell}(1,1,T'_\ell)-\frac{\psi_2(\lambda_2)[\lambda_2]}{\ell}(1,T'_\ell,1)+\frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1]\times[\lambda_2])(\ell+1)\bigg)(\mathcal{Z}^{(5)}_{\mu_3})\\
&=(\ell-1)\bigg(a_\ell(f)-\frac{\psi_1(\lambda_1)[\lambda_1]}{\ell}(\psi_2(\lambda_2)[\lambda_2]+\psi_2(\bar\lambda_2)[\bar\lambda_2])-(\psi_1(\lambda_1)[\lambda_1]+\psi_1(\bar\lambda_1)[\bar\lambda_1])\frac{\psi_2(\lambda_2)[\lambda_2]}{\ell}\\
&\quad+\frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1]\times[\lambda_2])(\ell+1)\bigg)(\mathcal{Z}^{(5)}_{\mu_3})\\
&=(\ell-1)\bigg(a_\ell(f)-\frac{\psi_1(\lambda_1)\psi_2(\bar\lambda_2)}{\ell}([\lambda_1]\times[\bar\lambda_2])-\frac{\psi_1(\bar\lambda_1)\psi_2(\lambda_2)}{\ell}([\bar\lambda_1]\times[\lambda_2])\\
&\quad+(1-\ell)\frac{\psi_1(\lambda_1)\psi_2(\lambda_2)}{\ell^2}([\lambda_1]\times[\lambda_2])\bigg)(\mathcal{Z}^{(5)}_{\mu_3}).
\end{aligned}$$

This implies the result via combining Theorem 2.2.2 and Lemma 3.1.2. □

Following [CD23, §2.2] verbatim, which borrows ideas from [DR17, §1.4], we can strip out the $(\ell-1)$ factor by quotienting out the diamond operators action and obtain modified classes

$$\mathcal{Z}^{(6)}_{\mu_3}\in H^1\big(K_0[\mu_3],T_f^\vee(\tilde\psi_1^{-1}\tilde\psi_2^{-1})(-1)\big).$$

Then the term in the right-hand side of Proposition 3.1.3 can be massaged to agree with the local Euler factor at $\mathcal{L}_2$ of the Galois representation $[T_f^\vee(\tilde\psi_1^{-1}\tilde\psi_2^{-1})(-1)]^\vee(1)=T_f(\tilde\psi_1\tilde\psi_2)(2)$, giving the correct norm relations:

Theorem 3.1.4. *Suppose $p\nmid 6h_{K_0}$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mu_3\in\mathcal{N}$ be a squarefree ideal of $\mathcal{O}_{K_3}$. Then there exists a collection of cohomology classes*

$$\mathcal{Z}_{\mu_3}\in H^1\big(K_0[\mu_3],T_f^\vee(\tilde\psi_1^{-1}\tilde\psi_2^{-1})(-1)\big)$$

such that for every split prime $\lambda_3\in\mathcal{L}$ of $\mathcal{O}_{K_3}$ of norm ℓ with $(\ell,m)=1$, we have the norm relation

$$\mathrm{Norm}^{K_0[\mu_3\,\lambda_3]}_{K_0[\mu_3]}(\mathcal{Z}_{\mu_3\,\lambda_3})=P_{\mathcal{L}_4}(\mathrm{Frob}_{\mathcal{L}_4/\,\lambda_3})(\mathcal{Z}_{\mu_3}),$$

where $P_{\mathcal{L}_4}(X)=\det(1-X\cdot\mathrm{Frob}_{\mathcal{L}_4/\,\lambda_3}\,|\,T_f(\tilde\psi_1\tilde\psi_2)(2))$.

Proof. The proof is parallel to the one of [CD23, Thm. 2.2.7]. First notes that $[\lambda_1]\times[\bar\lambda_2]$ corresponds to $\mathrm{Frob}_{\mathcal{L}_4/\,\lambda_3}\in H[\mu_3]^{(p)}$ under the map $e\circ w$ of Proposition 3.1.1. One then multiplies the class $\mathcal{Z}^{(6)}_{\mu_3\,\lambda_3}$ with $-\psi_1(\lambda_1)\psi_2(\bar\lambda_2)([\lambda_1]\times[\bar\lambda_2])$. From $\psi_1\psi_2((\ell))=\chi_{\psi_1}\chi_{\psi_2}(\ell)\ell^2=\epsilon_{K_1}(\ell)^{-1}\epsilon_{K_2}(\ell)^{-1}\ell^2=\ell^2$ since ℓ splits in K_0, $\psi_1(\lambda_1)\psi_2(\bar\lambda_2)=\tilde\psi_1(\mathcal{L}_4)\tilde\psi_2(\mathcal{L}_4)$, and the fact that $[\ell]\times[\ell]$ maps to the identity element inside the ring class group together with Lemma 9.6.1 and 9.6.3 in [Rub00], the result follows from the explicit formula of Proposition 3.1.3. □

3.2. **Construction for general weights and wild norm relations.** We now extend the above construction to other weights $(k,k_1,k_2)\in\mathbb{Z}^3_{\geq 1}$ following [CD23, Sec. 2.3]. Then we show that the constructed cohomology classes also satisfy the wild norm relations for the anticyclotomic $\mathbf{Z}_p^2$-extension of K_0.

First, we assume that $p\nmid 6h_{K_0}$. Assume further that p splits in K_0 i.e.

$$(p)=\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4\text{ in }K_0,$$

and $\mathcal{P}_2=\tau_3\mathcal{P}_1,\ \mathcal{P}_3=\tau_2\mathcal{P}_1,\ \mathcal{P}_4=\tau_1\mathcal{P}_1,$. Hence

$$(p)=\mathfrak{p}_1\bar{\mathfrak{p}}_1\text{ in }K_1,\ (p)=\mathfrak{p}_2\bar{\mathfrak{p}}_2\text{ in }K_2,$$

with $\mathcal{P}_1$ the prime of K_0 above p induced by our fixed embedding $i_p:\overline{\mathbf{Q}}\hookrightarrow\overline{\mathbf{Q}}_p$, and $\mathcal{P}_1$ lies above $\mathfrak{p}_i$, the prime of K_i for $i\in\{1,2\}$. Note that the numbering here is parallel to our convention in Section 3.0.1.

Let $\Gamma_{\mathfrak{p}_i}^{K_i}$ be the Galois group of the unique $\mathbf{Z}_p$-extension of K_i unramified outside $\mathfrak{p}_i$. There exists a unique Hecke character $\psi_{0,i}$ of K_i of infinity type $(-1,0)$ and conductor $\mathfrak{p}_i$ such that its p-adic avatar factors through $\Gamma_{\mathfrak{p}_i}^{K_i}$. The character ψ_i fixed at the beginning of this section can be decomposed as

$$\psi_i = \xi_i \psi_{0,i}^{k_i-1},$$

where ξ_i is a ray class character of K of conductor dividing $\mathfrak{f}_i\mathfrak{p}_i$. Noting that $\Gamma_{\mathfrak{p}_i}^{K_i}$ is a quotient of ${}^iH_{\mathfrak{f}_i\mathfrak{p}_i^\infty}$ allows us to view $\psi_{0,i}$ and ξ_i as characters of ${}^iH_{\mathfrak{f}_i\mathfrak{p}_i^\infty}$. The formal q-expansion

$$\boldsymbol{\theta}_{\xi_i}(q) = \sum_{(\mathfrak{a},\mathfrak{f}_i\mathfrak{p}_i)=1} \xi_i\psi_{0,i}(\mathfrak{a})[\mathfrak{a}]q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in \Lambda_{\mathfrak{p}_i}[\![q]\!],$$

where $\Lambda_{\mathfrak{p}_i} = \mathcal{O}[\![\Gamma_{\mathfrak{p}_i}^{K_i}]\!] \simeq \mathcal{O}[\![\Gamma]\!]$ and $\Gamma = 1+p\mathbf{Z}_p$, is the Hida family passing through θ_{ψ_i} (the specialisation of $\boldsymbol{\theta}_{\xi_i}$ at weight k_i and trivial character recovers the ordinary p-stabilization of θ_{ψ_i}). Here we identify $\Gamma_{\mathfrak{p}_i}^{K_i}$ with $1+p\mathbf{Z}_p$ via the (geometrically normalized) local Artin map.

Let $\boldsymbol{f}$ be the Hida family associated to f. Let $\boldsymbol{g} = \boldsymbol{\theta}_{\xi_1}$, $\boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}$ be the CM Hida families associated to ψ_1 and ψ_2, respectively. Denote by κ_f, κ_g, and κ_h the Dirichlet characters modulo p giving the p-part of the tame characters of $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$, respectively.

Under the assumption that $\xi_i\psi_{0,i} \not\equiv \omega \pmod{\mathfrak{P}}$ for $i \in \{1,2\}$, following equation (2.17) from [CD23, §2.3] and its notation, we have the $G_{\mathbf{Q}}$-equivariant maps

$$\begin{aligned} &H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_1}) \otimes \mathcal{O}[\![{}^1H^{(p)}_{\mathfrak{f}_1\mu_1\mathfrak{p}_1^\infty}]\!] \to \mathrm{Ind}_{K_1}^{\mathbf{Q}}\mathcal{O}_{(\xi_1\psi_{0,1})^{-1}}[{}^1H^{(p)}_{\mu_1}][\![\Gamma_{\mathfrak{p}_1}^{K_1}]\!], \\ &H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_2}) \otimes \mathcal{O}[\![{}^2H^{(p)}_{\mathfrak{f}_2\mu_2\mathfrak{p}_2^\infty}]\!] \to \mathrm{Ind}_{K_2}^{\mathbf{Q}}\mathcal{O}_{(\xi_2\psi_{0,2})^{-1}}[{}^2H^{(p)}_{\mu_2}][\![\Gamma_{\mathfrak{p}_2}^{K_2}]\!], \end{aligned} \tag{3.6}$$

where $\Gamma(m,p) = \Gamma_1(Nm)\cap\Gamma_0(p)$ is a congruence subgroup. Focusing on the class $\boldsymbol{\kappa}_m^{(2)}$ in equation (2.15) of op. cit., we first tensor it with $\mathcal{O}[{}^1H^{(p)}_{\mathfrak{f}_1\mu_1\mathfrak{p}_1^r}]$ and $\mathcal{O}[{}^2H^{(p)}_{\mathfrak{f}_2\mu_2\mathfrak{p}_2^r}]$, let $r\to\infty$, and then arrive at

$$\begin{aligned} \boldsymbol{\mathcal{Z}}^{(1)}_{\mu_3} \in H^1\big(\mathbf{Q}, H^1(\Gamma(1,p),\mathcal{D}'_{\kappa_f})\hat{\otimes}_{\mathcal{O}}(H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_g})\otimes\mathcal{O}[\![{}^1H^{(p)}_{\mathfrak{f}_1\mu_1\mathfrak{p}_1^\infty}]\!]) \\ \hat{\otimes}_{\mathcal{O}[D_m]}(H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_h})\otimes\mathcal{O}[\![{}^2H^{(p)}_{\mathfrak{f}_2\mu_2\mathfrak{p}_2^\infty}]\!])(2-\kappa^*_{\boldsymbol{fgh}})\big). \end{aligned}$$

Now choose a level-N test vector for f, denoted as $\breve{f}$. It comes with a specialization map

$$\pi_f : H^1(\Gamma(1,p),\mathcal{D}'_{\kappa_f})(1) \to T_f^\vee. \tag{3.7}$$

Under the natural maps induced by (3.6) and (3.7), the image of $\boldsymbol{\mathcal{Z}}^{(1)}_{\mu_3}$ is then

$$\boldsymbol{\mathcal{Z}}^{(2)}_{\mu_3} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} (\mathrm{Ind}_{K_1}^{\mathbf{Q}}\mathcal{O}_{(\xi_1\psi_{0,1})^{-1}}[{}^1H^{(p)}_{\mu_1}][\![\Gamma_{\mathfrak{p}_1}^{K_1}]\!])\hat{\otimes}_{\mathcal{O}[D_m]}(\mathrm{Ind}_{K_2}^{\mathbf{Q}}\mathcal{O}_{(\xi_2\psi_{0,2})^{-1}}[{}^2H^{(p)}_{\mu_2}][\![\Gamma_{\mathfrak{p}_2}^{K_2}]\!])(-1-\kappa^*_{\boldsymbol{fgh}})\big).$$

We first follow (3.3) and then apply the diagonal map $e\circ w$ in Proposition 3.1.1. This induces the following class

$$\boldsymbol{\mathcal{Z}}^{(3)}_{\mu_3} \in H^1\big(K_3, T_f^\vee(1-k/2)\otimes_{\mathcal{O}}\mathrm{Ind}_{K_0[\mu_3]}^{K_3}\Lambda_{\mathcal{O}}(\tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}\tilde{\kappa}_{\mathrm{ac},1}^{(k_1-2)/2}\tilde{\kappa}_{\mathrm{ac},2}^{(k_2-2)/2}\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})(1-(k_1+k_2)/2)\big). \tag{3.8}$$

Here, for $i\in\{1,2\}$, we identify $\Gamma_i^- = \mathrm{Gal}(K_{i,\infty}^-/K_i)$ with the anti-diagonal in $(1+p\mathbf{Z}_p)\times(1+p\mathbf{Z}_p) \simeq \mathcal{O}^{(1)}_{K_i,\mathfrak{p}_i}\times\mathcal{O}^{(1)}_{K_i,\bar{\mathfrak{p}}_i}$ via the geometric normalized Artin map, and define

$$\kappa_{ac,i} : \Gamma_i^- \to \mathbf{Z}_p^\times, \qquad ((1+p)^{-1/2},(1+p)^{1/2}) \mapsto (1+p)$$

(compare this with equation (2.19) of [CD23]). Note that here we identify the anticyclotomic $\mathbf{Z}_p$ extension of K_i with the unique $\mathbf{Z}_p$ extension of K_i unramified outside $\mathfrak{p}_i$ (projection to the anticyltomic part introduces a square root ($\gamma\mapsto\gamma^{(1-c)/2}$), see also [Hid15, p.636]).

We then identify the anticyclotomic $\mathbf{Z}_p^2$ extension $\Gamma^- = \mathrm{Gal}(K_{0,\infty}^-/K_0)$ of K_0 with $\Gamma_{\mathfrak{p}_1}^{K_1} \times \Gamma_{\mathfrak{p}_2}^{K_2}$ via the following diagram:

(3.9)

$$\begin{array}{ccccc} (\mathcal{O}^{(1)}_{K_0,\mathcal{P}_1} \times \mathcal{O}^{(1)}_{K_0,\mathcal{P}_2}) \times (\mathcal{O}^{(1)}_{K_0,\mathcal{P}_3} \times \mathcal{O}^{(1)}_{K_0,\mathcal{P}_4}) & \twoheadrightarrow & \frac{(1+p\mathbf{Z}_p)\times(1+p\mathbf{Z}_p)}{\mathrm{diag}} \times \frac{(1+p\mathbf{Z}_p)\times(1+p\mathbf{Z}_p)}{\mathrm{diag}} & \xrightarrow{\simeq} & \mathbf{Z}_p^2 \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{O}^{(1)}_{K_1,\mathfrak{p}_1} \times \mathcal{O}^{(1)}_{K_1,\bar{\mathfrak{p}}_1}) \times (\mathcal{O}^{(1)}_{K_2,\mathfrak{p}_2} \times \mathcal{O}^{(1)}_{K_2,\bar{\mathfrak{p}}_2}) & \twoheadrightarrow & \frac{(1+p\mathbf{Z}_p)\times(1+p\mathbf{Z}_p)}{\mathrm{diag}} \times \frac{(1+p\mathbf{Z}_p)\times(1+p\mathbf{Z}_p)}{\mathrm{diag}} & \xrightarrow{\simeq} & \mathbf{Z}_p \times \mathbf{Z}_p \end{array}$$

Let $\Lambda^- = \mathbf{Z}_p[\![\Gamma^-]\!]$ and define further $\boldsymbol{\kappa}_{ac} : \Gamma^- \to \Lambda^\times$ where

$$((1+p)^{-1/2}, (1+p)^{1/2}, (1+p)^{-1/2}, (1+p)^{1/2}) \mapsto [(1+p), (1+p)].$$

Given an $\mathcal{O}$-lattice T inside a G_{K_0}-representation V, Shapiro's lemma allows us to write

$$H^1\big(K_0, T\hat{\otimes}_{\mathcal{O}}\Lambda^-_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\big) \simeq H^1_{\mathrm{Iw}}(K_0[p^\infty], T),$$

where $H^1_{\mathrm{Iw}}(K_0[p^\infty], T) := \varprojlim_{r,s} H^1(K_0[\mathfrak{p}_3^r\bar{\mathfrak{p}}_3^s], T)$ with limit under the corestriction maps. Then the image of $\boldsymbol{\mathcal{Z}}^{(3)}_{\mu_3}$ in (3.8) under Shapiro's lemma is an Iwasawa cohomology class

$$\boldsymbol{\mathcal{Z}}_{\mu_3} \in H^1_{\mathrm{Iw}}\big(K_0[\mu_3 p^\infty], T_f^\vee(1-k/2) \otimes \tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}\tilde{\kappa}_{\mathrm{ac},1}^{(k_1-2)/2}\tilde{\kappa}_{\mathrm{ac},2}^{(k_2-2)/2}(1-(k_1+k_2)/2)\big) \tag{3.10}$$

for the conjugate self-dual representation $T_f^\vee(1-k/2)$ twisted by the Hecke character

$$\chi^{-1} = \tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}\mathbf{N}^{1-(k_1+k_2)/2}$$

(up to an anticyclotomic twist). Here χ is anticyclotomic and of infinity type (corresponding to the order $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$ or $(1, \tau_3, \tau_2, \tau_1)$):

$$\left(\frac{2-k_1-k_2}{2}, \frac{k_1+k_2-2}{2}, \frac{k_1-k_2}{2}, \frac{k_2-k_1}{2}\right).$$

Denote by

$$T_{f,\chi} = T_f^\vee(1-k/2) \otimes \chi^{-1}. \tag{3.11}$$

Following the proof of Theorem 3.1.4 and invoking [Rub00, Thm 6.4.1], we can obtain a collection of Iwasawa cohomology classes for anticyclotomic twists (to eliminate $\tilde{\kappa}_{\mathrm{ac},1}^{(k_1-2)/2}\tilde{\kappa}_{\mathrm{ac},2}^{(k_2-2)/2}$). We thus arrive at the proof for the wild norm relation, which is formulated inside the following theorem.

Theorem 3.2.1. *Suppose $p \nmid 6h_{K_0}$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mu_3 \in \mathcal{N}$ and denote $m = N_{K_3/\mathbf{Q}}(\mu_3)$. Then there exists a collection of Iwasawa cohomology classes*

$$\mathbf{z}_{f,\chi,\mu_3} \in H^1_{\mathrm{Iw}}\big(K_0[\mu_3 p^\infty], T_{f,\chi}\big)$$

such that for every split prime λ_3 of $\mathcal{O}_{K_3}$ of norm ℓ, where ℓ splits in K_0, with $(\ell, mp) = 1$ we have the norm relation

$$\mathrm{Norm}_{K_0[\mu_3]}^{K_0[\mu_3\lambda_3]}(\mathbf{z}_{f,\chi,\mu_3\lambda_3}) = P_{\mathcal{L}_4}(\mathrm{Frob}_{\mathcal{L}_4})(\mathbf{z}_{f,\chi,\mu_3}),$$

where $P_{\mathcal{L}_4}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathcal{L}_4} | (T_{f,\chi})^\vee(1))$.

4. Selmer groups

In this section, we show that the classes constructed in Theorem 3.2.1 land in certain Selmer groups defined by Greenberg [Gre94]. Keeping the setup at the start of Section 3, we further assume that f is a p-ordinary newform of even weight $k \geq 2$ with $p \nmid N_f$.

Let χ be an anticyclotomic Hecke character of K_0 of infinity type $(-a, a, -b, b)$ for some integers $a, b \geq 0$. We will focus on the conjugate self-dual G_{K_0}-representation

$$V_{f,\chi} := V_f^\vee(1-k/2) \otimes \chi^{-1}.$$

Definition 4.0.1. For each prime $\mathcal{P} \in \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4\}$ of K_0 above p, we fix a $G_{K_{0,\mathcal{P}}}$-stable subspace $\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}) \subset V_{f,\chi}$ and denote

$$\mathscr{F}^-_{\mathcal{P}}(V_{f,\chi}) = V_{f,\chi}/\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}).$$

Let L be a finite extension of K_0. The *Greenberg Selmer group* $\mathrm{Sel}_{\mathscr{F}}(L, V_{f,\chi})$ attached to $\mathscr{F} = \{\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi})\}_{\mathcal{P}|p}$ is defined by

$$\mathrm{Sel}_{\mathscr{F}}(L, V_{f,\chi}) := \ker\left\{ H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_{\mathscr{F}}(L_w, V_{f,\chi})} \right\}, \tag{4.1}$$

where w runs over the finite primes of L, and the local conditions are given by

$$H^1_{\mathscr{F}}(L_w, V_{f,\chi}) = \begin{cases} \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\} & \text{if } w \nmid p, \\ \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, \mathscr{F}^-_{\mathcal{P}}(V_{f,\chi}))\} & \text{if } w \mid \mathcal{P} \mid p. \end{cases}$$

We fix a lattice $T_{f,\chi} \subset V_{f,\chi}$. Let $H^1_{\mathscr{F}}(L_w, T_{f,\chi})$ be the inverse image of $H^1_{\mathscr{F}}(L_w, V_{f,\chi})$ under the natural map

$$H^1(L_w, T_{f,\chi}) \to H^1(L_w, V_{f,\chi}).$$

This then defines $\mathrm{Sel}_{\mathscr{F}}(L, T_{f,\chi})$ as in (4.1). For any $\mathbf{Z}_p^2$-extension $L_\infty = \bigcup_{r,s} L_{r,s}$ of L, we put

$$\mathrm{Sel}_{\mathscr{F}}(L_\infty, T_{f,\chi}) := \varprojlim_{r,s} \mathrm{Sel}_{\mathscr{F}}(L_{r,s}, T_{f,\chi}),$$

where the inverse limit is taken with respect to the corestriction map. We also put $\mathrm{Sel}_{\mathscr{F}}(L_\infty, V_{f,\chi}) := \mathrm{Sel}_{\mathscr{F}}(L_\infty, T_{f,\chi}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Note that this group is independent of the chosen lattice $T_{f,\chi}$.

Definition 4.0.2. We also define the *Bloch-Kato Selmer group* $\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi})$ following [BK90]:

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) := \ker\left\{ H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_f(L_w, V_{f,\chi})} \right\},$$

where the local conditions are given by

$$H^1_f(L_w, V_{f,\chi}) = \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\},$$

at primes $w \nmid p$, and the crystalline condition at primes $w \mid p$:

$$H^1_f(L_w, V_{f,\chi}) = \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, V_{f,\chi} \otimes \mathbf{B}_{\mathrm{cris}})\}$$

with $\mathbf{B}_{\mathrm{cris}}$ being Fontaine's crystalline period ring. The local conditions $H^1_f(L_w, T_{f,\chi}) \subset H^1(L_w, T_{f,\chi})$ are defined by propagation similarly.

Besides the crystalline condition, there are three local conditions at primes $\mathcal{P} \mid p$ that we will be interested in:

(1) The **strict** condition:
$$\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}) = 0$$
(2) The **relaxed** condition:
$$\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}) = V_{f,\chi}$$
(3) The **ordinary** condition, corresponding to the fact that the restriction of $V_{f,\chi}$ to $G_{\mathbf{Q}_p}$ is reducible (see equation 2.1):
$$\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi}) = V^+_{f,\chi} := V_f^{\vee,+}(1 - k/2) \otimes \chi^{-1}$$

Definition 4.0.3. Denote by $\mathrm{Sel}_{\alpha,\beta,\gamma,\delta}(K_0, V)$ the subgroup of $H^1(K_0, V)$ where classes are unramified at all primes $v \nmid p$; and they satisfy the conditions α, β, γ, δ at $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ respectively, where $\alpha, \beta, \gamma, \delta \in \{\mathrm{rel}, \mathrm{str}, \mathrm{ord}\}$, and these conditions correspond to the relaxed, strict, and ordinary condition respectively.

We will now compute the explicit local conditions for the Bloch-Kato Selmer group. Here we shall adopt the convention that the p-adic cyclotomic character has Hodge–Tate weight -1. Thus, since χ has infinity type $(-a, a, -b, b)$, the p-adic avatar of χ has Hodge–Tate weight $a, -a, b, -b$ at $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ respectively.

Lemma 4.0.4. *Assume that $a \geq b$. For any finite extension L of K_0 we have*

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) = \begin{cases} \mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(L, V_{f,\chi}) & \text{if } k \geq 2a+2, \\ \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(L, V_{f,\chi}) & \text{if } 2b+2 \leq k < 2a+2 \\ \mathrm{Sel}_{\mathrm{rel,str,rel,str}}(L, V_{f,\chi}) & \text{if } k < 2b+2. \end{cases}$$

Proof. By the Panchiskin condition [BK90, Thm 4.1(ii)] (see also [Nek00, (3.1)-(3.2)] and [Fla90, Lem. 2, p. 125]), for every prime $w|\mathcal{P}|p$ of $L/K_0/\mathbf{Q}$ we have

$$H^1_f(L_w, V_{f,\chi}) = \mathrm{im}\{H^1(L_w, \mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi})) \to H^1(L_w, V_{f,\chi})\},$$

where $\mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi}) \subset V_{f,\chi}$ is a $G_{K_{\mathcal{P}}}$-stable subspace (assuming it exists) such that the Hodge–Tate weights of $\mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi})$ (resp. $V_{f,\chi}/\mathrm{Fil}^1_{\mathcal{P}}(V_{f,\chi})$) are all < 0 (resp. ≥ 0).

Now, by computing the Hodge–Tate weights table of $V^+_{f,\chi}$ and $V^-_{f,\chi} := V_{f,\chi}/V^+_{f,\chi}$ at the primes of K_0 above p:

	$V^+_{f,\chi}$	$V^-_{f,\chi}$
HT weight at $\mathcal{P}_1$	$-a-k/2$	$-a-1+k/2$
HT weight at $\mathcal{P}_2$	$a-k/2$	$a-1+k/2$
HT weight at $\mathcal{P}_3$	$-b-k/2$	$-b-1+k/2$
HT weight at $\mathcal{P}_4$	$b-k/2$	$b-1+k/2$

we obtained the equalities in the lemma. □

Fix a choice of Galois stable subgroups $\mathscr{F} = \{\mathscr{F}^+_{\mathcal{P}}(V_{f,\chi})\}_{\mathcal{P}|p}$ and let

$$A_{f,\chi} := \mathrm{Hom}_{\mathbf{Z}_p}(T_{f,\chi}, \mu_{p^\infty}).$$

Define the associated *dual Selmer group* $\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi})$ by

$$\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi}) := \ker\left\{H^1(L, A_{f,\chi}) \to \prod_w \frac{H^1(L_w, A_{f,\chi})}{H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})}\right\},$$

where $H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})$ is the orthogonal complement of $H^1_{\mathscr{F}}(L_w, T_{f,\chi})$ under local Tate pairing

$$H^1(L_w, T_{f,\chi}) \times H^1(L_w, A_{f,\chi}) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

One can then compute the following:

(1) The dual Selmer group of $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(L, T_{f,\chi})$ consists of classes that are unramified outside p and have the strict, relaxed, ordinary, ordinary condition at $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ respectively. Compatibly with Definition 4.0.3, this can be denoted as $\mathrm{Sel}_{\mathrm{str,rel,ord,ord}}(L, A_{f,\chi})$.

(2) The dual Selmer group of $\mathrm{Sel}_{\mathrm{rel,str,rel,str}}(L, T_{f,\chi})$ consists of classes that are unramified outside p and have the strict, relaxed, strict, relaxed condition at $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ respectively. Compatibly with Definition 4.0.3, this can be denoted as $\mathrm{Sel}_{\mathrm{str,rel,str,rel}}(L, A_{f,\chi})$.

(3) The dual Selmer group of $\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(L, T_{f,\chi})$ consists of classes that are unramified outside p, and land in the image of the natural map

$$H^1(L_w, \mathscr{F}^+_{\mathcal{P}}(A_{f,\chi})) \to H^1(L_w, A_{f,\chi}), \quad \mathscr{F}^+_{\mathcal{P}}(A_{f,\chi}) := \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{F}^-_{\mathcal{P}}(T_{f,\chi}), \mu_{p^\infty}),$$

for $w|\mathcal{P}|p$. Compatibly with Definition 4.0.3, this can be denoted as $\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(L, A_{f,\chi})$.

5. Triple product p-adic L-function and Selmer group

Here, we will recall some conventions on Hida families, triple product p-adic L-function ($\boldsymbol{f}$-unbalanced) and Selmer groups (balanced and $\boldsymbol{f}$-unbalanced) following [Hsi21].

5.1. **Hida families.** We follow the convention of [Hsi21, §3.1]. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbf{Q}_p$. Let $\mathbb{I}$ be a normal domain, finite flat over the Iwasawa algebra

$$\Lambda := \mathcal{O}[\![1 + p\mathbf{Z}_p]\!].$$

Let N be a positive integer primes p and $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character. Denote by $S^o(N, \chi, \mathbb{I}) \subset \mathbb{I}[\![q]\!]$ the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and branch character χ.

Let $\mathfrak{X}_{\mathbb{I}}^+ \subset \operatorname{Spec} \mathbb{I}(\overline{\mathbf{Q}}_p)$ be the set of arithmetic points of $\mathbb{I}$, which consists of the ring homomorphisms $Q : \mathbb{I} \to \overline{\mathbf{Q}}_p$ such that for some $k_Q \in \mathbf{Z}_{\geq 2}$ called the weight of Q and $\epsilon_Q(z) \in \mu_{p^\infty}$,

$$Q|_{1+p\mathbf{Z}_p} : z \mapsto z^{k_Q - 1}\epsilon_Q(z).$$

We say that $\boldsymbol{f} = \sum_{n=1}^{\infty} a_n(\boldsymbol{f}) q^n \in S^o(N, \chi, \mathbb{I})$ is a primitive Hida family if the specialization $\boldsymbol{f}_Q$ for every $Q \in \mathfrak{X}_{\mathbb{I}}^+$ gives the q-expansion of an ordinary p-stabilised newform of weight k_Q and tame conductor N. Let $\mathfrak{X}_{\mathbb{I}}^{\mathrm{cls}} \subset \operatorname{Spec} \mathbb{I}(\overline{\mathbf{Q}}_p)$ be the set of ring homomorphisms Q as above with $k_Q \in \mathbf{Z}_{\geq 1}$ such that $\boldsymbol{f}_Q$ is the q-expansion of a classical modular form.

Given $\boldsymbol{f}$ a primitive Hida family of tame conductor N, one can associate a Galois representation

$$\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \operatorname{Aut}_{\mathbb{I}}(V_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(\mathbb{I}),$$

where the determinant of $\rho_{\boldsymbol{f}}$ is $\chi_{\mathbb{I}} \cdot \varepsilon_{\mathrm{cyc}}$, see [Hsi21, §3.2]. By [Wil88, Thm. 2.2.2], the restriction of $V_{\boldsymbol{f}}$ to $G_{\mathbf{Q}_p}$ is reducible and one has a short exact sequence

$$0 \to V_{\boldsymbol{f}}^+ \to V_{\boldsymbol{f}} \to V_{\boldsymbol{f}}^- \to 0.$$

Here the quotient $V_{\boldsymbol{f}}^-$ is free of rank one over $\mathbb{I}$, with $G_{\mathbf{Q}_p}$ acting via the unramified character sending an arithmetic Frobenius Frob_p^{-1} to $a_p(\boldsymbol{f})$. Let $\mathbb{T}(N, \mathbb{I})$ be the Hecke algebra acting on $\bigoplus_\chi S^o(N, \chi, \mathbb{I})$, where χ runs over the characters of $(\mathbf{Z}/Np\mathbf{Z})^\times$. There is a $\mathbb{I}$-algebra homomorphism attached to $\boldsymbol{f}$

$$\lambda_{\boldsymbol{f}} : \mathbb{T}(N, \mathbb{I}) \to \mathbb{I}$$

that factors through a local component $\mathbb{T}_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal containing $\ker \lambda_{\boldsymbol{f}}$. Following [Hid88], we define the congruence ideal $C(\boldsymbol{f})$ of $\boldsymbol{f}$ by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\operatorname{Ann}_{\mathbb{T}_{\mathfrak{m}}}(\ker \lambda_{\boldsymbol{f}})) \subset \mathbb{I}.$$

Under the assumption that the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished, Wiles [Wil95] and Hida [Hid88] prove that $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}} \in \mathbb{I}$.

5.2. **CM Hida families revisited.** We explicitly construct CM Hida families, following the exposition in [Hsi21, §8.1]. Let K be an imaginary quadratic field of discriminant $-D_K < 0$, and suppose that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$.

Let K_∞ be the $\mathbf{Z}_p^2$-extension of K. Let $K(\mathfrak{p}^\infty)$ be the maximal subfield of K_∞ unramified outside $\mathfrak{p}$. Put

$$\Gamma_\infty := \operatorname{Gal}(K_\infty/K) \simeq \mathbf{Z}_p^2, \qquad \Gamma_{\mathfrak{p}} := \operatorname{Gal}(K(\mathfrak{p}^\infty)/K) \simeq \mathbf{Z}_p.$$

For every ideal $\mathfrak{c} \subset \mathcal{O}_K$, recall that $K_{\mathfrak{c}}$ is the ray class field of K of conductor $\mathfrak{c}$. Using our notation, $K(\mathfrak{p}^\infty)$ is the maximal $\mathbf{Z}_p$-extension of K inside $K_{\mathfrak{p}^\infty}$. Denote by $\mathrm{Art}_{\mathfrak{p}}$ the restriction of the Artin map to $K_{\mathfrak{p}}^\times$, with geometric normalisation. Then $\mathrm{Art}_{\mathfrak{p}}$ induces an embedding $1 + p\mathbf{Z}_p \to \Gamma_{\mathfrak{p}}$, where we identified $\mathbf{Z}_p^\times$ and $\mathcal{O}_{K_{\mathfrak{p}}}^\times$ via $\imath_p$. Let $\gamma_{\mathfrak{p}}$ be the image of $1 + p$ hence it will be a topological generator of $\Gamma_{\mathfrak{p}}$.

For each variable S let $\Psi_S : \Gamma_\infty \to \mathcal{O}[\![S]\!]^\times$ be the universal character given by

$$\Psi_S(\sigma) = (1 + S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbf{Z}_p$ is such that $\sigma|_{K(\mathfrak{p}^\infty)} = \gamma_{\mathfrak{p}}^{l(\sigma)}$. Now assume that $\mathfrak{c}$ is prime to p. Given a finite order character $\xi : G_K \to \mathcal{O}^\times$ of conductor dividing $\mathfrak{c}$, let

$$\boldsymbol{\theta}_\xi(S)(q) = \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{c})=1} \xi(\sigma_{\mathfrak{a}}) \Psi^{-1}_{\frac{1+S}{1+p}-1}(\sigma_{\mathfrak{a}}) q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in \mathcal{O}[\![S]\!][\![q]\!],$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(K_{\mathfrak{c}\mathfrak{p}^\infty}/K)$ is the Artin symbol of $\mathfrak{a}$. Then $\boldsymbol{\theta}_\xi(S)$ is a Hida family defined over $\mathcal{O}[\![S]\!]$ of tame level $N_{K/\mathbf{Q}}(\mathfrak{c})D_K$ and tame character $(\xi \circ \mathscr{V})\epsilon_K\omega^{-1}$, where $\mathscr{V} : G^{\mathrm{ab}}_{\mathbf{Q}} \to G^{\mathrm{ab}}_K$ is the transfer map and ϵ_K is the quadratic character corresponding to $K/\mathbf{Q}$.

5.3. **Triple products of Hida families.** Let

$$\boldsymbol{f} \in S^o(N_f, \chi_f, \mathbb{I}_f), \quad \boldsymbol{g} \in S^o(N_g, \chi_g, \mathbb{I}_g), \quad \boldsymbol{h} \in S^o(N_h, \chi_h, \mathbb{I}_h)$$

be three primitive Hida families such that

$$\chi_f\chi_g\chi_h = \omega^{2a} \text{ for some } a \in \mathbf{Z}, \tag{5.1}$$

where ω is the Teichmüller character. Let

$$\mathcal{R} = \mathbb{I}_f \hat{\otimes}_{\mathcal{O}} \mathbb{I}_g \hat{\otimes}_{\mathcal{O}} \mathbb{I}_h$$

be a finite extension of the three-variable Iwasawa algebra $\Lambda\hat{\otimes}_{\mathcal{O}}\Lambda\hat{\otimes}_{\mathcal{O}}\Lambda$.

Let $\mathfrak{X}^+_{\mathcal{R}} \subset \mathrm{Spec}\,\mathcal{R}(\overline{\mathbf{Q}}_p)$ be the weight space of $\mathcal{R}$ given by

$$\mathfrak{X}^+_{\mathcal{R}} := \left\{ \underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}^+_{\mathbb{I}_f} \times \mathfrak{X}^{\mathrm{cls}}_{\mathbb{I}_g} \times \mathfrak{X}^{\mathrm{cls}}_{\mathbb{I}_h} \ : \ k_{Q_1} + k_{Q_2} + k_{Q_3} \equiv 0 \pmod 2 \right\}.$$

One can then partition $\mathfrak{X}^+_{\mathcal{R}} = \mathfrak{X}^{\mathrm{bal}}_{\mathcal{R}} \sqcup \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} \sqcup \mathfrak{X}^{\boldsymbol{g}}_{\mathcal{R}} \sqcup \mathfrak{X}^{\boldsymbol{h}}_{\mathcal{R}}$ as follows:

(1) the set of *balanced* weights:

$$\mathfrak{X}^{\mathrm{bal}}_{\mathcal{R}} := \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_1} + k_{Q_2} + k_{Q_3} > 2k_{Q_i} \text{ for all } i \in \{1,2,3\} \right\},$$

(2) the set of $\boldsymbol{f}$-unbalanced weights:

$$\mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} := \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_1} \geq k_{Q_2} + k_{Q_3} \right\},$$

(3) the set of $\boldsymbol{g}$-unbalanced weights:

$$\mathfrak{X}^{\boldsymbol{g}}_{\mathcal{R}} := \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_2} \geq k_{Q_1} + k_{Q_3} \right\},$$

(4) the set of $\boldsymbol{h}$-unbalanced weights:

$$\mathfrak{X}^{\boldsymbol{h}}_{\mathcal{R}} := \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_3} \geq k_{Q_1} + k_{Q_2} \right\}.$$

Let $\mathbf{V} = V_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}$ be the triple tensor product Galois representation attached to $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. By (5.1), one can decompose the determinant of $\mathbf{V}$ as $\det \mathbf{V} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$. Put

$$\mathbf{V}^\dagger := \mathbf{V} \otimes \mathcal{X}^{-1}. \tag{5.2}$$

This is a self-dual twist of $\mathbf{V}$. For any $\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}}$, denote by $\mathbf{V}^\dagger_{\underline{Q}}$ the corresponding specialisation.

For each prime ℓ, let $\varepsilon_\ell(\mathbf{V}^\dagger_{\underline{Q}})$ be the epsilon factor attached to the local representation $\mathbf{V}^\dagger_{\underline{Q}}|_{G_{\mathbf{Q}_\ell}}$ (cf. [Tat79, p. 21]). We assume that for some $\underline{Q} \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}}$, we have

$$\varepsilon_\ell(\mathbf{V}^\dagger_{\underline{Q}}) = +1 \text{ for all prime factors } \ell \text{ of } N_f N_g N_h. \tag{5.3}$$

Note that condition (5.3) is independent of $\underline{Q}$ (see [Hsi21, §1.2]). Furthermore it implies that the sign of the functional equation for the triple product L-function (with center at $s = 0$)

$$L(\mathbf{V}^\dagger_{\underline{Q}}, s)$$

is $+1$ (resp. -1) for all $\underline{Q} \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} \cup \mathfrak{X}^{\boldsymbol{g}}_{\mathcal{R}} \cup \mathfrak{X}^{\boldsymbol{h}}_{\mathcal{R}}$ (resp. $\underline{Q} \in \mathfrak{X}^{\mathrm{bal}}_{\mathcal{R}}$).

Theorem 5.3.1 (Theorem A in [Hsi21])**.** *Let $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$ be three primitive Hida families satisfying conditions (5.1) and (5.3). Assume also that $\gcd(N_f,N_g,N_h)$ is squarefree, and the residual representation $\bar\rho_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished. Fix a generator $\eta_{\boldsymbol{f}}$ of the congruence ideal of $\boldsymbol{f}$. Then there exists a unique element*

$$\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\in\mathcal{R}$$

such that for all $\underline{Q}=(Q_1,Q_2,Q_3)\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ of weight (k_1,k_2,k_3) with $\epsilon_{Q_1}=1$ we have

$$\big(\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(\underline{Q})\big)^2=\Gamma_{\mathbf{V}^\dagger_{\underline{Q}}}(0)\cdot\frac{L(\mathbf{V}^\dagger_{\underline{Q}},0)}{(\sqrt{-1})^{2k_1}\cdot\Omega^2_{\boldsymbol{f}_{Q_1}}}\cdot\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger_{\underline{Q}}))\cdot\prod_{q\in\Sigma_{\mathrm{exc}}}(1+q^{-1})^2,$$

where:

- $\Gamma_{\mathbf{V}^\dagger_{\underline{Q}}}(0)=16(2\pi)^{-2k_1}\Gamma(w_{\underline{Q}})\Gamma(w_{\underline{Q}}+2-k_2-k_3)\Gamma(w_{\underline{Q}}+1-k_2)\Gamma(w_{\underline{Q}}+1-k_3),$
$$\text{and}\qquad w_{\underline{Q}}=(k_1+k_2+k_3-2)/2;$$
- $\Omega_{\boldsymbol{f}_{Q_1}}$ *is the Hida canonical period*
$$\Omega_{\boldsymbol{f}_{Q_1}}:=(-2\sqrt{-1})^{k_1+1}\cdot\frac{\|\boldsymbol{f}^\circ_{Q_1}\|^2_{\Gamma_0(N_f)}}{\eta_{\boldsymbol{f}_{Q_1}}}\cdot\Big(1-\frac{\chi_f'(p)p^{k_1-1}}{\alpha^2_{Q_1}}\Big)\Big(1-\frac{\chi_f'(p)p^{k_1-2}}{\alpha^2_{Q_1}}\Big),$$
with $\boldsymbol{f}^\circ_{Q_1}\in S_{k_1}(\Gamma_0(N_f))$ the newform of conductor N_f associated with $\boldsymbol{f}_{Q_1}$, χ_f' the prime-to-p part of χ_f, and α_{Q_1} the specialisation of $a_p(\boldsymbol{f})\in\mathbb{I}_f^\times$ at Q_1;
- $\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger_{\underline{Q}}))$ *is the modified p-Euler factor and Σ_{exc} is an explicitly defined subset of the prime factors of $N_fN_gN_h$,* [Hsi21, p. 416].

5.4. **Triple product Selmer groups.** Recall from equation (5.2) that $\mathbf{V}^\dagger=\mathbf{V}\otimes\mathcal{X}^{-1}$ is the self-dual twist of the Galois representation associated to a triple of primitive Hida families $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ given (5.1).

Definition 5.4.1. Let

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger):=\big(V_{\boldsymbol{f}}\otimes V_{\boldsymbol{g}}^+\otimes V_{\boldsymbol{h}}^++V_{\boldsymbol{f}}^+\otimes V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}^++V_{\boldsymbol{f}}^+\otimes V_{\boldsymbol{g}}^+\otimes V_{\boldsymbol{h}}\big)\otimes\mathcal{X}^{-1},$$

and define the balanced local condition $\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^\dagger):=\mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger))\to\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)\big).$$

Similarly, let

$$\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger)=\big(V_{\boldsymbol{f}}^+\otimes V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}\big)\otimes\mathcal{X}^{-1},$$
$$\mathscr{F}_p^{\boldsymbol{h}}(\mathbf{V}^\dagger)=\big(V_{\boldsymbol{f}}\otimes V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}^+\big)\otimes\mathcal{X}^{-1}.$$

Define the $\boldsymbol{f}$-unbalanced local condition $\mathrm{H}^1_{\boldsymbol{f}}(\mathbf{Q}_p,\mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\boldsymbol{f}}(\mathbf{Q}_p,\mathbf{V}^\dagger):=\mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger))\to\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)\big)$$

and the $\boldsymbol{h}$-unbalanced local condition $\mathrm{H}^1_{\boldsymbol{h}}(\mathbf{Q}_p,\mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\boldsymbol{h}}(\mathbf{Q}_p,\mathbf{V}^\dagger):=\mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^{\boldsymbol{h}}(\mathbf{V}^\dagger))\to\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)\big).$$

Note that the maps appearing in these definitions are injective, so we can identify $\mathrm{H}^1_\star(\mathbf{Q}_p,\mathbf{V}^\dagger)$ with $\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^\star(\mathbf{V}^\dagger))$ for $\star\in\{\mathrm{bal},\boldsymbol{f},\boldsymbol{h}\}$.

Definition 5.4.2. Let $\star\in\{\mathrm{bal},\boldsymbol{f},\boldsymbol{h}\}$. Define the Selmer group $\mathrm{Sel}^\star(\mathbf{Q},\mathbf{V}^\dagger)$ by

$$\mathrm{Sel}^\star(\mathbf{Q},\mathbf{V}^\dagger):=\ker\Big\{\mathrm{H}^1(\mathbf{Q},\mathbf{V}^\dagger)\to\frac{\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)}{\mathrm{H}^1_\star(\mathbf{Q}_p,\mathbf{V}^\dagger)}\times\prod_{v\neq p}\mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}},\mathbf{V}^\dagger)\Big\}.$$

We call $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger)$ the balanced Selmer group, $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{V}^\dagger)$ the $\boldsymbol{f}$-unbalanced Selmer group, and $\mathrm{Sel}^{\boldsymbol{h}}(\mathbf{Q},\mathbf{V}^\dagger)$ the $\boldsymbol{h}$-unbalanced Selmer group.

Definition 5.4.3. Let $\mathbf{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger, \mu_{p^\infty})$ and let $\star \in \{\mathrm{bal}, \boldsymbol{f}, \boldsymbol{h}\}$. Define $\mathrm{H}^1_\star(\mathbf{Q}_p, \mathbf{A}^\dagger) \subset \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)$ to be the orthogonal complement of $\mathrm{H}^1_\star(\mathbf{Q}_p, \mathbf{V}^\dagger)$ under the local Tate pairing

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger) \times \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Similarly as above, we then define the balanced, the $\boldsymbol{f}$-unbalanced and the $\boldsymbol{h}$-unbalanced Selmer groups with coefficients in $\mathbf{A}^\dagger$ by

$$\mathrm{Sel}^\star(\mathbf{Q}, \mathbf{A}^\dagger) := \ker\left\{\mathrm{H}^1(\mathbf{Q}, \mathbf{A}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)}{\mathrm{H}^1_\star(\mathbf{Q}_p, \mathbf{A}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}}, \mathbf{A}^\dagger)\right\}.$$

Let

$$X^\star(\mathbf{Q}, \mathbf{A}^\dagger) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}^\star(\mathbf{Q}, \mathbf{A}^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$$

be the Pontryagin dual of $\mathrm{Sel}^\star(\mathbf{Q}, \mathbf{A}^\dagger)$.

6. Arithmetic applications

Finally, we obtain some arithmetic results from our constructed Euler system through identifying our classes as an anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner [JNS] and the explicit reciprocity law.

6.1. **Reciprocity law and Greenberg–Iwasawa main conjectures.** Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families as in §5.1 satisfying (5.1). Let $N = \mathrm{lcm}(N_f, N_g, N_h)$. The big diagonal class constructed in [BSV22, §8.1]

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{H}^1(\mathbf{Q}, \mathbf{V}^\dagger(N)), \tag{6.1}$$

where $\mathbf{V}^\dagger(N)$ (this is $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ using the notation of [BSV22]) is a free $\mathcal{R}$-module isomorphic to finitely many copies of $\mathbf{V}^\dagger$, can be identified with classes $\widetilde{\boldsymbol{\kappa}}_m^{(1)}$, $\boldsymbol{\kappa}_m^{(2)}$ in equation (2.14), (2.15) respectively of [CD23]. The definition of the Selmer groups in §5.4 extends to $\mathbf{V}^\dagger(N)$, and by [BSV22, Cor. 8.2] we have $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger(N))$. Now we choose level-$N$ test vectors $(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$, provided by [Hsi21, Thm. A], to project the classes from $\mathbb{V}^\dagger(N)$ to $\mathbb{V}^\dagger$.

We define more $G_{\mathbf{Q}_p}$-invariant subspaces of $\mathbf{V}^\dagger$:

$$\begin{aligned}
\mathscr{F}_p^3(\mathbf{V}^\dagger) &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1},\\
\mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} &= V_{\boldsymbol{f}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1},\\
\mathbf{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1},\\
\mathbf{V}_{\boldsymbol{h}}^{\boldsymbol{fg}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^- \otimes \mathcal{X}^{-1},
\end{aligned} \tag{6.2}$$

and obtain

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)/\mathscr{F}_p^3(\mathbf{V}^\dagger) \cong \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} \oplus \mathbf{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} \oplus \mathbf{V}_{\boldsymbol{h}}^{\boldsymbol{fg}}, \tag{6.3}$$

Assume that the congruence ideal $C(\boldsymbol{f}) \subset \mathbb{I}_f$ is principal, generated by the nonzero $\eta_{\boldsymbol{f}} \in \mathbb{I}_f$ (this will be satisfied when the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished). One can deduce from results in [KLZ17] the construction of an injective three-variable p-adic regulator map with pseudo-null cokernel:

$$\mathrm{Log}^{\eta_{\boldsymbol{f}}} : \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}) \to \mathcal{R}, \tag{6.4}$$

see the explicit map in [CD23, §4.3.1] and the explanation in [BSV22, §7.3].

Let $\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}}$ be the image of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under the natural composition of maps:

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger) \xrightarrow{\mathrm{res}_p} \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)) \to \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)/\mathscr{F}_p^3(\mathbf{V}^\dagger)) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}), \tag{6.5}$$

where we first restrict at p and then project onto the first direct summand in (6.3). The following result is an explicit reciprocity law that relates diagonal cycles with the triple product p-adic L-functions.

Theorem 6.1.1 (Theorem A in [BSV22]). *Let $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ be a triple of primitive Hida families as in Theorem 5.3.1. Then*

$$\mathrm{Log}^{\eta_{\boldsymbol{f}}}(\mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))_{\boldsymbol{f}}) = \mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}).$$

Assume that the associated ring $\mathcal{R}$ is regular. Similar to [ACR25, §7.3], the following result can be seen as the equivalence between two different formulations of the Iwasawa main conjecture in the style of Greenberg [Gre94] for the p-adic deformation $\mathbf{V}^\dagger$, relating the $\boldsymbol{f}-$unbalanced Selmer group to the balanced one (or one with $\mathscr{L}_p^{\boldsymbol{f}}$ and another 'without p-adic L-functions').

Proposition 6.1.2 (Proposition 4.3.3 in[CD23]). *The following statements* (I) *and* (II) *are equivalent:*

(I) *$\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is nonzero, the modules $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{V}^\dagger)$ and $X^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}^\dagger)$ are both $\mathcal{R}$-torsion, and*

$$\mathrm{char}_{\mathcal{R}}\big(X^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}^\dagger)\big) = \big(\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^2\big)$$

in $\mathcal{R}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$.

(II) *$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is not $\mathcal{R}$-torsion, the modules $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger)$ and $X^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}^\dagger)$ have both $\mathcal{R}$-rank one, and*

$$\mathrm{char}_{\mathcal{R}}\big(X^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}^\dagger)_{\mathrm{tors}}\big) = \mathrm{char}_{\mathcal{R}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger)}{\mathcal{R}\cdot\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})}\right)^2$$

in $\mathcal{R}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$, where the subscript tors *denotes the $\mathcal{R}$-torsion submodule.*

6.2. **The set-up.** Let $f\in S_{2r}(pN_f)$ be a p-stabilised newform, and suppose the residual representation $\bar\rho_f$ is absolutely irreducible and p-distinguished. By Hida theory, f is the specialisation of a unique primitive Hida family $\boldsymbol{f}\in S^o(N_f,\mathbb{I})$ at an arithmetic point $Q_1\in\mathfrak{X}_{\mathbb{I}}^+$ of weight $2r$. For $i\in\{1,2\}$ let $\mathfrak{f}_i\subset\mathcal{O}_{K_i}$ be an ideal coprime to pN_f, ξ_i be ray class characters of K_i of conductors dividing $\mathfrak{f}_i$. Let χ_{ξ_i} be the central character of ξ_i. We assume that

$$\chi_{\xi_1}\epsilon_{K_1}\chi_{\xi_2}\epsilon_{K_2}=1, \tag{6.6}$$

and let

$$\boldsymbol{g}_1=\boldsymbol{\theta}_{\xi_1}(S_1)\in\mathcal{O}[\![S_1]\!][\![q]\!],\quad \boldsymbol{g}_2=\boldsymbol{\theta}_{\xi_2}(S_2)\in\mathcal{O}[\![S_2]\!][\![q]\!] \tag{6.7}$$

be the CM Hida families attached to ξ_1 and ξ_2, respectively.

The triple $(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{g}_2)$ satisfies conditions (5.1) and the associated $\boldsymbol{f}$-unbalanced triple product p-adic L-function $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g}_1,\boldsymbol{g}_2)$ is an element in $\mathcal{R}=\mathbb{I}\hat\otimes_{\mathcal{O}}\mathcal{O}[\![S_1]\!]\hat\otimes_{\mathcal{O}}\mathcal{O}[\![S_2]\!]\simeq\mathbb{I}[\![S_1,S_2]\!]$. Let

$$\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f,\boldsymbol{g}_1,\boldsymbol{g}_2)\in\mathcal{O}[\![S_1,S_2]\!] \tag{6.8}$$

be its image under the natural map $\mathbb{I}[\![S_1,S_2]\!]\to\mathcal{O}[\![S_1,S_2]\!]$ defined by Q_1.

Write $\mathbf{V}^\dagger_{Q_1}$ for the specialisation of $\mathbf{V}^\dagger$ at Q_1. Let $V_f^\vee$ be the Galois representation associated to f, and recall that $\det(V_f^\vee)=\varepsilon_{\mathrm{cyc}}^{2r-1}$ in our conventions. Setting $T_i=\mathbf{v}^{-1}(1+S_i)-1$ $(i\in\{1,2\})$, we have $\det(V_{\boldsymbol{g}_{T_1}}\otimes V_{\boldsymbol{h}_{T_2}})=\Psi_{T_1}\Psi_{T_2}\circ\mathscr{V}$, and so

$$\begin{aligned}\mathbf{V}^\dagger_{Q_1}&\simeq T_f^\vee\otimes(\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\Psi_{T_1})\otimes(\mathrm{Ind}_K^{\mathbf{Q}}\xi_2^{-1}\Psi_{T_2})\otimes\varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2}\circ\mathscr{V})\\&\simeq T_f^\vee(1-r)\otimes\mathrm{Ind}_{K_0}^{\mathbf{Q}}\tilde\xi_1^{-1}\tilde\Psi_{T_1}\tilde\xi_2^{-1}\tilde\Psi_{T_2},\end{aligned} \tag{6.9}$$

where $T_f^\vee$ is a $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice inside $V_f^\vee$. In particular, we get

$$\mathrm{H}^1(\mathbf{Q},\mathbf{V}^\dagger_{Q_1})\simeq\mathrm{H}^1(K_0,T_f^\vee(1-r)\otimes\tilde\xi_1^{-1}\tilde\Psi_{T_1}\tilde\xi_2^{-1}\tilde\Psi_{T_2}) \tag{6.10}$$

by Shapiro's lemma.

6.3. **Local conditions at p of the Euler system.** Recall from Theorem 3.2.1 that we have constructed classes

$$\mathbf{z}_{f,\chi,\mu_3} \in H^1_{\mathrm{Iw}}(K_0[\mu_3 p^\infty], T_{f,\chi}),$$

where $T_{f,\chi} = T_f^\vee(1-r) \otimes \chi^{-1}$ and $\chi^{-1} = \tilde{\psi}_1^{-1}\tilde{\psi}_2^{-1}\mathbf{N}^{1-(k_1+k_2)/2}$.

Proposition 6.3.1. *Suppose $p \nmid 6h_{K_0}$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mu_3 \in \mathcal{N}$ (taken from Section 3.0.1) run over squarefree product of prime ideals of $\lambda_3 \in \mathcal{L}$ with $m = N_{K_3/\mathbf{Q}}(\mu_3)$ coprime to p. The class $\mathbf{z}_{f,\chi,\mu_3}$ of Theorem 3.2.1 satisfies*

$$\mathbf{z}_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3 p^\infty], T_{f,\chi}).$$

Proof. By [BSV22, Cor. 8.2] and [CD23, Sec. 3.2], the class $\mathbf{z}_{f,\chi,\mu_3}$ lands in the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger)$, where the balanced local condition at p upon specialised to f is given by

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_1}) \simeq \left(T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-\mathbf{c}}\tilde{\Psi}^{\mathbf{c}}_{T_2}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \tilde{\xi}_1^{-\mathbf{c}}\tilde{\Psi}^{\mathbf{c}}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}\right). \tag{6.11}$$

Put $\widetilde{\mathbf{V}}^\dagger_{Q_1} = T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}$ then Shapiro Lemma tells us that $H^1(\mathbf{Q}, \mathbf{V}^\dagger_{Q_1}) \simeq H^1(K_0, \widetilde{\mathbf{V}}^\dagger_{Q_1})$, see (6.10). Following [CD23, Sec. 5.3], the local condition $\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_1})$ cutting out the specialised balanced Selmer group at p corresponds to

$$\begin{aligned}
&\mathscr{F}^{\mathrm{bal}}_{\mathcal{P}_1}(\widetilde{\mathbf{V}}^\dagger_{Q_1}|_{G_{K_0}}) = T_f^\vee(1-k/2) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2},\\
&\mathscr{F}^{\mathrm{bal}}_{\mathcal{P}_2}(\widetilde{\mathbf{V}}^\dagger_{Q_1}|_{G_{K_0}}) = 0,\\
&\mathscr{F}^{\mathrm{bal}}_{\mathcal{P}_3}(\widetilde{\mathbf{V}}^\dagger_{Q_1}|_{G_{K_0}}) = T_f^{\vee,+}(1-k/2) \otimes \tilde{\xi}_1^{-\mathbf{c}}\tilde{\Psi}^{\mathbf{c}}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2},\\
&\mathscr{F}^{\mathrm{bal}}_{\mathcal{P}_4}(\widetilde{\mathbf{V}}^\dagger_{Q_1}|_{G_{K_0}}) = T_f^{\vee,+}(1-k/2) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-\mathbf{c}}\tilde{\Psi}^{\mathbf{c}}_{T_2}.
\end{aligned} \tag{6.12}$$

Hence the class $\mathbf{z}_{f,\chi,\mu_3}$ satisfies the relaxed-strict-ordinary-ordinary condition at the primes above p.

On the other hand, at the primes $w \nmid p$, because $V_{f,\chi}$ is conjugate self-dual and pure of weight -1, we see that

$$H^0(K_0[\mu_3\mathfrak{p}_3^r\bar{\mathfrak{p}}_3^s]_w, V_{f,\chi}) = H^2(K_0[\mu_3\mathfrak{p}_3^r\bar{\mathfrak{p}}_3^s]_w, V_{f,\chi}) = 0$$

for all r, s, and therefore

$$H^1(K_0[\mu_3\mathfrak{p}_3^r\bar{\mathfrak{p}}_3^s]_w, V_{f,\chi}) = 0$$

by Tate's local Euler characteristic formula. This implies the torsionness of $H^1(K_0[\mu_3\mathfrak{p}_3^r\bar{\mathfrak{p}}_3^s]_w, T_{f,\chi})$, and one has the following inclusion:

$$\mathrm{res}_w(\mathbf{z}_{f,\chi,\mu_3}) \in \varprojlim_{r,s} H^1_f(K_0[\mu_3\mathfrak{p}_3^r\bar{\mathfrak{p}}_3^s]_w, T_{f,\chi}),$$

which concludes the proof. ☐

Proposition 6.3.2. *Via the isomorphism (6.10),*

(1) *the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_1})$ can be rewritten as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}),$$

(2) *the $\boldsymbol{f}$-unbalanced Selmer group $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_1})$ can be rewritten as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}).$$

(3) *the $\boldsymbol{h}$-unbalanced Selmer group $\mathrm{Sel}^{\boldsymbol{h}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_1})$ can be rewritten as*

$$\mathrm{Sel}^{\boldsymbol{h}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_1}) \simeq \mathrm{Sel}_{\mathrm{rel,str,rel,str}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}).$$

Proof. For the balanced case, see Proposition 6.3.1. For the $\boldsymbol{f}$-unbalanced case, note that

$$\begin{aligned}\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{Q_1}^\dagger) \simeq \big(T_f^{\vee,+}(1-r)\otimes\tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}\big)\oplus\big(T_f^{\vee,+}(1-r)\otimes\tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-\mathbf{c}}\tilde{\Psi}_{T_2}^{\mathbf{c}}\big)\\ \oplus\big(T_f^{\vee,+}(1-r)\otimes\tilde{\xi}_1^{-\mathbf{c}}\tilde{\Psi}_{T_1}^{\mathbf{c}}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}\big)\oplus\big(T_f^{\vee,+}(1-r)\otimes\tilde{\xi}_1^{-\mathbf{c}}\tilde{\Psi}_{T_1}^{\mathbf{c}}\tilde{\xi}_2^{-\mathbf{c}}\tilde{\Psi}_{T_2}^{\mathbf{c}}\big).\end{aligned} \tag{6.13}$$

and the result follows.

The $\boldsymbol{h}$-unbalanced case can be obtained in a similar manner where:

$$\mathscr{F}_p^{\boldsymbol{h}}(\mathbf{V}_{Q_1}^\dagger) \simeq \big(T_f^{\vee}(1-r)\otimes\tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}\big)\oplus\big(T_f^{\vee}(1-r)\otimes\tilde{\xi}_1^{-\mathbf{c}}\tilde{\Psi}_{T_1}^{\mathbf{c}}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}\big).$$

□

As a consequence, we also obtain the following isomorphisms for the Selmer groups with coefficients in $\mathbf{A}_{Q_1}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}_{Q_1}^\dagger,\mu_{p^\infty})$ by local Tate duality. Let $A_f(r) = \mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r),\mu_{p^\infty})$.

Corollary 6.3.3. *We can identify the balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}_{Q_1}^\dagger)$ *as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}_{Q_1}^\dagger)\simeq \mathrm{Sel}_{\mathrm{str,rel,ord,ord}}(K_0, A_f(r)\otimes\tilde{\xi}_1\tilde{\Psi}_{T_1}^{-1}\tilde{\xi}_2\tilde{\Psi}_{T_2}^{-1}),$$

the $\boldsymbol{f}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}_{Q_1}^\dagger)$ *as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}_{Q_1}^\dagger)\simeq \mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_0, A_f(r)\otimes\tilde{\xi}_1\tilde{\Psi}_{T_1}^{-1}\tilde{\xi}_2\tilde{\Psi}_{T_2}^{-1}),$$

and the $\boldsymbol{h}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{h}}(\mathbf{Q},\mathbf{A}_{Q_1}^\dagger)$ *as*

$$\mathrm{Sel}^{\boldsymbol{h}}(\mathbf{Q},\mathbf{A}_{Q_1}^\dagger)\simeq \mathrm{Sel}_{\mathrm{str,rel,str,rel}}(K_0, A_f(r)\otimes\tilde{\xi}_1\tilde{\Psi}_{T_1}^{-1}\tilde{\xi}_2\tilde{\Psi}_{T_2}^{-1}).$$

6.4. **Applying the general machinery.** We show some arithmetic applications by invoking the general Euler system machinery of Jetchev–Nekovář–Skinner [JNS], see some details for the imaginary quadratic case in [Do22, §4.3] and [ACR25, §8]. These results will be used to deduce the Bloch–Kato conjecture and the anticyclotomic Iwasawa main conjecture by exploiting the relation between our Euler system classes and special values of complex and p-adic L-functions via an explicit reciprocity law.

For every ideal $\mu_3\in\mathcal{N}$, denote by

$$z_{f,\chi,\mu_3}\in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3],T_{f,\chi})$$

the image of $\mathbf{z}_{f,\chi,\mu_3}$ from Theorem 3.2.1 under the projection

$$\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3p^\infty],T_{f,\chi})\to \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3],T_{f,\chi}). \tag{6.14}$$

And denote the base class

$$z_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(z_{f,\chi,1})\in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0,T_{f,\chi}).$$

Note that since we assume $p\nmid h_{K_0}$, $K_0[1]$ is actually the same with K_0 (recall that $K_0[\mathfrak{n}]$ is the maximal p-extension inside the ring class field of K_0 of conductor $\mathfrak{n}$). Therefore, $z_{f,\chi}=z_{f,\chi,1}$.

Theorem 6.4.1. *Assume that f is not of CM-type, non-Eisenstein at $\mathfrak{P}$, and that $p\nmid 6h_{K_0}$. One has:*

$$z_{f,\chi}\text{ is non-torsion}\quad\Rightarrow\quad \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0,V_{f,\chi})\text{ is one-dimensional.}$$

Proof. Combining Theorem 3.2.1 and Proposition 6.3.1, the system of classes

$$\big\{z_{f,\chi,\mu_3}\in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3],T_{f,\chi})\ :\ \mu_3\in\mathcal{N}\big\} \tag{6.15}$$

forms an anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner [JNS] for the relaxed-strict-ordinary-ordinary Greenberg Selmer group.

Under the assumption that f is not of CM-type, the following properties (i)–(iii) follow from Momose's big image results [Mom81] as in [LLZ15, Prop. 7.1.4]:

(i) $V_{f,\chi}$ is absolutely irreducible;
(ii) There is an element $\sigma\in G_{K_0}$ fixing $K_0[1]K_0(\mu_{p^\infty},(\mathcal{O}_{K_0}^\times)^{1/p^\infty})$ such that $V_{f,\chi}/(\sigma-1)V_{f,\chi}$ is one-dimensional;

(iii) There is an element $\gamma \in G_{K_0}$ fixing $K_0[1]K_0(\mu_{p^\infty}, (\mathcal{O}_{K_0}^\times)^{1/p^\infty})$ such that $V_{f,\chi}^{\gamma=1} = 0$.

Hence, the fact that $z_{f,\chi}$ is non-torsion implies the one-dimensionality of $\mathrm{Sel}_{\mathrm{rel},\mathrm{str},\mathrm{ord},\mathrm{ord}}(K_0, V_{f,\chi})$ by the general machinery of [JNS]. □

Recall that $K_{0,\infty}^-$ is the anticyclotomic $\mathbf{Z}_p^2$ extension over K_0 and $\Lambda_{K_0}^- = \mathbf{Z}_p[\![\mathrm{Gal}(K_{0,\infty}^-/K_0)]\!]$. Let $\mathbf{z}_{f,\chi,1}$ be the $\Lambda_{K_0}^-$-adic class of Theorem 3.2.1 of conductor $\mu_3 = (1)$, and put the Iwasawa-theoretic base class

$$\mathbf{z}_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(\mathbf{z}_{f,\chi,(1)}).$$

Again $\mathbf{z}_{f,\chi} = \mathbf{z}_{f,\chi,(1)}$ from the assumption on the class number of K_0. Note that by Proposition 6.3.1, one has

$$\mathbf{z}_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, T_{f,\chi}).$$

Definition 6.4.2. We say that f has big image at $\mathfrak{P}$ if the image of $G_{\mathbf{Q}}$ in $\mathrm{Aut}_{\mathcal{O}}(T_f^\vee)$ contains a conjugate of $\mathrm{SL}_2(\mathbf{Z}_p)$.

Remark 6.4.3. By a theorem of Ribet [Rib85], if f is not of CM-type then it has big image for all but finitely many primes of L.

Denote by

$$X_{\mathrm{str},\mathrm{rel},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, A_{f,\chi}) = \mathrm{Hom}_{\mathbf{Z}_p}\big(\varinjlim \mathrm{Sel}_{\mathrm{str},\mathrm{rel},\mathrm{ord},\mathrm{ord}}(K_0[\mathfrak{p}_3^r\bar{\mathfrak{p}}_3^s], A_{f,\chi}), \mathbf{Q}_p/\mathbf{Z}_p\big).$$

One then has a divisibility towards an anticyclotomic Iwasawa main conjecture 'without p-adic L-functions' as follows:

Theorem 6.4.4. *Assume that f is not of CM-type, has big image at $\mathfrak{P}$, and that $p \nmid 6h_{K_0}$. If $\mathbf{z}_{f,\chi}$ is non-torsion, then:*

1. $X_{\mathrm{str},\mathrm{rel},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, A_{f,\chi})$ *and* $\mathrm{Sel}_{\mathrm{rel},\mathrm{str},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, T_{f,\chi})$ *both have* $\Lambda_{K_0}^-$*-rank one.*
2. *And we have the divisibility*

$$\mathrm{char}_{\Lambda_{K_0}^-}(X_{\mathrm{str},\mathrm{rel},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, A_{f,\chi})_{\mathrm{tors}}) \supset \mathrm{char}_{\Lambda_{K_0}^-}\left(\frac{\mathrm{Sel}_{\mathrm{rel},\mathrm{str},\mathrm{ord},\mathrm{ord}}(K_{0,\infty}^-, T_{f,\chi})}{\Lambda_{K_0}^- \cdot \mathbf{z}_{f,\chi}}\right)^2$$

in $\Lambda_{K_0}^-$.

Here, the subscript tors *denotes the* $\Lambda_{K_0}^-$*-torsion submodule.*

Proof. Combining Theorem 3.2.1 and Proposition 6.3.1, the system of classes

$$\big\{\mathbf{z}_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str},\mathrm{ord},\mathrm{ord}}(K_0[\mu_3 p^\infty], T_{f,\chi}) :\ \mu_3 \in \mathcal{N}\big\} \tag{6.16}$$

forms a $\Lambda_{K_0}^-$-adic anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner for the relaxed-strict-ordinary-ordinary Selmer group.

Under the assumption that f has big image at $\mathfrak{P}$, the following properties hold (see [LLZ15, Prop. 7.1.6])

(i) $\bar{T}_{f,\chi} := T_{f,\chi}/\mathfrak{P}T_{f,\chi}$ is absolutely irreducible;

(ii) There is an element $\sigma \in G_K$ fixing $K_0[1]K_0(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $T_{f,\chi}/(\sigma-1)T_{f,\chi}$ is free of rank 1 over $\mathcal{O}$;

(iii) There is an element $\gamma \in G_{K_0}$ fixing $K_0[1]K_0(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ and acting as multiplication by a scalar $a_\gamma \neq 1$ on $\bar{T}_{f,\chi}$;

and so the non-torsionness of $\mathbf{z}_{f,\chi}$ implies the conclusions by the general machinery of [JNS]. □

6.5. **On the Bloch–Kato conjecture in rank 0.** Our first application is the Bloch–Kato conjecture in analytic rank zero for the conjugate self-dual G_{K_0}-representation $V_{f,\chi} = V_f^\vee(1-r)\otimes\chi^{-1}$.

Assumption 6.1. We assume that the anticyclotomic Hecke character χ over K_0 can be decomposed as:

$$\chi = \tilde{\psi}_1\tilde{\psi}_2\mathbf{N}^{(k_1+k_2-2)/2},$$

where

(1) ψ_1 is a Hecke character of K_1 of infinity type $(1-k_1, 0)$, with $k_1 \geq 1$, and modulus $\mathfrak{f}_1$.
(2) ψ_2 is a Hecke character of K_2 of infinity type $(1-k_2, 0)$, with $k_2 \geq 1$ and modulus $\mathfrak{f}_2$.
(3) $\tilde{\psi}_i$ is the Hecke character of K_0, obtained by composing $\mathbb{A}_{K_0}^\times \xrightarrow{\mathbb{N}_{K_0/K_i}} \mathbb{A}_{K_i}^\times \xrightarrow{\psi_i} \mathbb{C}$ for each $i \in \{1,2\}$.
(4) By swapping K_1 and K_2, we may assume that $k_2 \geq k_1$.

In this scenario, the infinity type of χ (corresponding to the order $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$ or $(1, \tau_3, \tau_2, \tau_1)$) is

$$\left(\frac{2-k_1-k_2}{2}, \frac{k_1+k_2-2}{2}, \frac{k_1-k_2}{2}, \frac{k_2-k_1}{2}\right).$$

Theorem 6.5.1. *Let $f \in S_k(\Gamma_0(pN_f))$ be a p-ordinary p-stabilised newform of weight $k = 2r \geq 2$ which is old at p. Let χ be an anticyclotomic Hecke character of K_0 as in (6.1). Assume that:*

(1) *Either $k \geq k_1 + k_2$ or $k_2 - k_1 \geq k$;*
(2) *$N_f\mathcal{O}_{K_3} = \mathfrak{n}^+\mathfrak{n}^-$ where $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) is divisible only by primes which are split (respectively inert) in K_0/K_3 and $\mathfrak{n}^-$ is a squarefree product of an even number of primes.*
(3) *$\bar{\rho}_f$ is absolutely irreducible;*
(4) *$(pN_f, \mathrm{Norm}_{K_1/\mathbf{Q}}(\mathfrak{f}_1)\mathrm{Norm}_{K_2/\mathbf{Q}}(\mathfrak{f}_2)D_{K_0}) = 1$;*
(5) *$p \nmid 6h_{K_0}$, the class number of K_0;*

then we have the following implication

$$L(f/K_0, \chi, r) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) = 0.$$

In other words, the Bloch–Kato conjecture holds in analytic rank zero for $V_{f,\chi}$.

Proof. We consider the CM Hida families

$$\boldsymbol{g} = \boldsymbol{\theta}_{\xi_1}(S_1), \quad \boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}(S_2),$$

that pass through θ_{ψ_1} and θ_{ψ_2} respectively. Note that the triple $(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2)$ satisfies (5.3). Then the isomorphism (6.9) of the associated $\mathbf{V}^\dagger_{Q_1}$ together with the specialization $\mathcal{Q}_1$ corresponding to θ_{ψ_1} and θ_{ψ_2} show that

$$L(\mathbb{V}^\dagger_{\mathcal{Q}_1}, 0) = L(f/K_0, \chi, r).$$

By Theorem 6.1.1 we then have

$$L(f/K_0, \chi, r) \neq 0 \quad \Longrightarrow \quad \mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}, \mathcal{Q}_1} \neq 0.$$

From our construction and Proposition 6.3.2, the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_{\mathcal{Q}_1} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, V_{f,\chi})$ is the base class of the anticyclotomic Euler system

$$\left\{z_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[m], T_{f,\chi}) : \ \mu_3 \in \mathcal{N}\right\}$$

of (6.15). Recall again that $K_0[1] = K_0$ from the assumption $p \nmid h_{K_0}$. By Theorem 6.4.1, we conclude that the Selmer group $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, V_{f,\chi})$ is one-dimensional, spanned by

$$z_{f,\chi} = \mathrm{Norm}_{K_0}^{K_0[1]}(z_{f,\chi,1}) = \kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_{\mathcal{Q}_1}.$$

If $k \geq k_1 + k_2$, we observe that the composition of maps in 6.5 corresponds to the composition of

$$\begin{aligned}\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_f^\vee(1-r)\otimes\tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) &\xrightarrow{\mathrm{res}_{\mathcal{P}_1}} \mathrm{H}^1(K_{0,\mathcal{P}_1}, T_f^\vee(1-r)\otimes\tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2}) \\ &\longrightarrow \mathrm{H}^1(K_{0,\mathcal{P}_1}, T_f^{\vee,-}(1-r)\otimes\tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2})\end{aligned}$$

by equation (6.10). Hence $\mathrm{res}_{\mathcal{P}_1}(z_{f,\chi}) \neq 0$ by the reciprocity law Theorem 6.1.1. The vanishing of $\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_0, V_{f,\chi})$ then follows by a standard argument using Poitou–Tate duality (see [Do22, §5.1.1]). This yields the result by using the Lemma 4.0.4 for $k \geq k_1 + k_2$ to identify the latter group with $\mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi})$.

If $k_2 - k_1 \geq k$, similarly by using $\mathrm{res}_{\mathcal{P}_2}(z_{f,\chi}) \neq 0$ we obtain the vanishing of $\mathrm{Sel}_{\mathrm{rel,str,rel,str}}(K_0, V_{f,\chi})$, which is again the Bloch-Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi})$ by Lemma 4.0.4 for $k_2 - k_1 \geq k$. □

Remark 6.5.2. Let $\epsilon(f,\chi)$ to be the sign of the functional equation for $V_{f,\chi}$. Then $\epsilon(f,\chi) = \prod \epsilon(\pi_{K_{0,v}} \otimes \chi_v, 1/2)$ over places v of K_0 as a product of local root numbers. If $v|\mathfrak{n}^+$ then $\epsilon(\pi_{K_{0,v}} \otimes \chi_v, 1/2) = +1$ and if $v|\mathfrak{n}^-$ then $\epsilon(\pi_{K_{0,v}} \otimes \chi_v, 1/2) = -1$. Therefore the contribution from the local places is $+1$ due to assumption (2). At the infinity places,

$$\begin{aligned}\epsilon_\infty(\pi_{K_0} \otimes \chi, \frac{1}{2}) &= i^{|k-1+(k_1+k_2-2)|+|k-1-(k_1+k_2-2)|+|k-1+(k_2-k_1)|+|k-1-(k_2-k_1)|} \\ &= \begin{cases} +1 & \text{if} \quad k > (k_1+k_2-2) \\ -1 & \text{if} \quad k_2 - k_1 < k \leq k_1 + k_2 - 2 \\ +1 & \text{if} \quad k \leq k_2 - k_1. \end{cases}\end{aligned}$$

Hence conditions (1) and (2) of Theorem 6.5.1 then imply that $\epsilon(f,\chi) = 1$.

6.6. **On the Iwasawa main conjecture.** Our second application is an evidence towards the anticyclotomic Iwasawa main conjecture for modular forms. Recall that we have an eigenform f of weight $k = 2r \geq 2$ with trivial nebentypus and an anticyclotomic character χ satisfying Assumption 6.1. Let

$$A_{f,\chi} = \mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r) \otimes \chi^{-1}, \mu_{p^\infty}).$$

Theorem 6.6.1. *Under the same assumption as in Theorem 6.5.1, we assume further that:*

1. *$\bar{\rho}_f$ is p-distinguished,*
2. *f has big image,*
3. *$p > k-2$.*

If $k \geq k_1 + k_2$ then $\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_{0,\infty}^-, A_{f,\chi})$ is cotorsion over $\Lambda_{K_0}^-$. Furthermore, inside $\Lambda_{K_0}^- \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, we have the following inclusion

$$\mathrm{char}_{\Lambda_{K_0}^-}\big(\mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_{0,\infty}^-, A_{f,\chi})^\vee\big) \supset \big(\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)^2\big).$$

Proof. Recall from Corollary 6.3.3 that we have

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger) \simeq \mathrm{Sel}_{\mathrm{ord,ord,ord,ord}}(K_{0,\infty}^-, A_f(r) \otimes \chi), \tag{6.17}$$

where $\mathbb{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbb{V}^\dagger, \mu_{p^\infty})$.

Note that from (6.8), $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)$ is an element of $\mathcal{O}[\![S_1, S_2]\!]$. We then identify $\Lambda_{K_0}^- \simeq \mathcal{O}[\![S_1, S_2]\!]$ via the diagram (3.9). The p-adic L-function $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)$ is nonzero by [Hun17, Thm. C]. Note that our assumption that $k \geq k_1 + k_2$ ensures that we are in the critical specializations i.e. $-k_\sigma/2 < m_\sigma < k_\sigma/2$ for all $\sigma \in \Sigma$, following the notation of op. cit.. Hence Theorem 6.1.1 and the proof of Theorem 6.5.1 implies that the class

$$\kappa(f, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_f^\vee(1-r) \otimes \tilde{\xi}_1^{-1}\tilde{\Psi}_{T_1}\tilde{\xi}_2^{-1}\tilde{\Psi}_{T_2})$$

is non-torsion. By construction, we can treat $\kappa(f, \boldsymbol{g}, \boldsymbol{h})$ as the base class of the $\Lambda_{K_0}^-$-adic anticyclotomic Euler system

$$\big\{\mathbf{z}_{f,\chi,\mu_3} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0[\mu_3 p^\infty], T_{f,\chi}) : \ \mu_3 \in \mathcal{N}\big\}$$

in (6.16). Via the isomorphism 6.10, the result follows immediately from Theorem 6.4.4 applied to

$$\mathbf{z}_{f,\chi} := \mathrm{Norm}_{K_0}^{K_0[1]}(\mathbf{z}_{f,\chi,(1)}) = \kappa(f, \boldsymbol{g}, \boldsymbol{h}), \tag{6.18}$$

the equivalence between two different formulation of the Iwasawa main conjecture in Proposition 6.1.2, and the Selmer group isomorphism (6.17). □

Remark 6.6.2. Within Theorem 6.6.1, the RHS can be compared to the p-adic L-function of Wan [Wan15, Thm. 86] and Hung [Hun17] (under assumptions [Fuj06, Thm. 11.1,11.2] and [Wan15, Thm. 103]). The author then expects that the full Iwasawa Main Conjecture, which means an equality of Theorem 6.6.1, will follow by combining Theorem 6.6.1 with the opposite inclusion of Wan [Wan15] and the vanishing of the μ-invariant of the p-adic L-function [Hun17] (those are generalizations of Skinner-Urban [SU14] and Vatsal [Vat03]).

Remark 6.6.3. One expects a similar result that if $k_2 - k_1 \geq k$ then $\mathrm{Sel}_{\mathrm{str,rel,str,rel}}(K_\infty^-, A_{f,\chi})$ is cotorsion over $\Lambda_{K_0}^-$ together with the following inclusion inside $\Lambda_{K_0}^- \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$,

$$\mathrm{char}_{\Lambda_{K_0}^-}\left(\mathrm{Sel}_{\mathrm{str,rel,str,rel}}(K_{0,\infty}^-, A_{f,\chi})^\vee\right) \supset \left(\mathscr{L}_p^{\boldsymbol{h},\eta_{\boldsymbol{h}}}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)^2\right).$$

The only missing ingredient is the non-vanishing of the p-adic L-function in this region.

6.7. **On the Bloch–Kato conjecture in rank 1.** Our last application is extracted from the proof of Theorem 6.6.1. It provides a result towards the Bloch–Kato conjecture in rank 1.

Theorem 6.7.1. *Under the same assumption as in Theorem 6.6.1, if $k_1 + k_2 - 2 \geq k \geq k_2 - k_1 + 2$ (which induces $L(f/K, \chi, r) = 0$), then*

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) \geq 1.$$

Proof. The class $\mathbf{z}_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})$ is non-torsion via the proof of Theorem 6.6.1. Furthermore, $\mathbf{z}_{f,\chi}$ is the base of a $\Lambda_{K_0}^-$-adic anticyclotomic Euler system as in (6.18) for the relaxed-strict-ordinary-ordinary Selmer group. Theorem 6.4.4 then implies that $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})$ has $\Lambda_{K_0}^-$-rank 1. The natural map (compare with the projection (6.14))

$$\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_{0,\infty}^-, T_{f,\chi})/(\gamma_{1,-} - 1, \gamma_{2,-} - 1) \to \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_{f,\chi}) \tag{6.19}$$

is injective (see also [MR04, Prop. 5.3.14] and [Gre99, p. 72]).

Hence, the Selmer group $\mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_{f,\chi})$ has a positive $\mathcal{O}$-rank. The theorem then follows by Lemma 4.0.4, which computes the local conditions of the Bloch-Kato Selmer group explictly. □

Remark 6.7.2. Note that by letting $z_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel,str,ord,ord}}(K_0, T_{f,\chi})$ be the image of $\mathbf{z}_{f,\chi}$ under the map (6.19), such a class $z_{f,\chi} \in \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi})$ satisfies:

$$z_{f,\chi} \neq 0 \quad \Longrightarrow \quad \dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K_0, V_{f,\chi}) = 1.$$

by Theorem 6.4.1.

References

[ACR25] Raúl Alonso, Francesc Castella, and Óscar Rivero, *An anticyclotomic Euler system for adjoint modular Galois representations*, Ann. Inst. Fourier (Grenoble) **75** (2025), no. 1, 291–329.

[BD05] Massimo Bertolini and Henri Darmon, *Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions*, Ann. of Math. (2) **162** (2005), no. 1, 1–64.

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *Generalized Heegner cycles and p-adic Rankin L-series*, Duke Math. J. **162** (2013), no. 6, 1033–1148.

[BDR$^+$22] Massimo Bertolini, Henri Darmon, Victor Rotger, Marco Adamo Seveso, and Rodolfo Venerucci, *Heegner points, Stark-Heegner points, and diagonal classes*, Astérisque (2022), no. 434, xviii+201.

[BK90] Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.

[BSV22] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci, *Reciprocity laws for balanced diagonal classes*, Astérisque (2022), no. 434, 77–174.

[Cas20] Francesc Castella, *On the p-adic variation of Heegner points*, J. Inst. Math. Jussieu **19** (2020), no. 6, 2127–2164.

[CD23] Francesc Castella and Kim Tuan Do, *Diagonal cycles and anticyclotomic Iwasawa theory of modular forms*, 2023, available at https://arxiv.org/abs/2303.06751.

[CH15] Masataka Chida and Ming-Lun Hsieh, *On the anticyclotomic Iwasawa main conjecture for modular forms*, Compos. Math. **151** (2015), no. 5, 863–897.

[CH18] Francesc Castella and Ming-Lun Hsieh, *Heegner cycles and p-adic L-functions*, Math. Ann. **370** (2018), no. 1-2, 567–628.

[Chi17] Masataka Chida, *Selmer groups and central values of L-functions for modular forms*, Ann. Inst. Fourier (Grenoble) **67** (2017), no. 3, 1231–1276.

[Do22] Kim Tuan Do, *Construction of an anticyclotomic Euler system with applications*, Ph.D. thesis, Princeton University, 2022.

[DR17] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-functions*, J. Amer. Math. Soc. **30** (2017), no. 3, 601–672.

[Fla90] Matthias Flach, *A generalisation of the Cassels-Tate pairing*, J. Reine Angew. Math. **412** (1990), 113–127.

[FT91] Albrecht Fröhlich and Martin Taylor, *Algebraic number theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1991.

[Fuj06] Kazuhiro Fujiwara, *Deformation rings and Hecke algebras in the totally real case*, 2006, available at https://arxiv.org/abs/math/0602606.

[GK92] Benedict Gross and Stephen Kudla, *Heights and the central critical values of triple product L-functions*, Compositio Math. **81** (1992), no. 2, 143–209.

[Gre94] Ralph Greenberg, *Iwasawa theory and p-adic deformations of motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 193–223.

[Gre99] ———, *Iwasawa theory for elliptic curves*, Arithmetic theory of elliptic curves (Cetraro, 1997), Lecture Notes in Math., vol. 1716, Springer, Berlin, 1999, pp. 51–144.

[GS95] Benedict Gross and Chad Schoen, *The modified diagonal cycle on the triple product of a pointed curve*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 3, 649–679.

[GZ86] Benedict Gross and Don Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320.

[Hid88] Haruzo Hida, *Modules of congruence of Hecke algebras and L-functions associated with cusp forms*, Amer. J. Math. **110** (1988), no. 2, 323–382.

[Hid15] ———, *Big Galois representations and p-adic L-functions*, Compos. Math. **151** (2015), no. 4, 603–664.

[Hsi21] Ming-Lun Hsieh, *Hida families and p-adic triple product L-functions*, Amer. J. Math. **143** (2021), no. 2, 411–532.

[Hun17] Pin-Chi Hung, *On the non-vanishing* $\bmod \ell$ *of central L-values with anticyclotomic twists for Hilbert modular forms*, J. Number Theory **173** (2017), 170–209.

[JNS] Dimitar Jetchev, Jan Nekovář, and Christopher Skinner, preprint.

[KLZ17] Guido Kings, David Loeffler, and Sarah Livia Zerbes, *Rankin-Eisenstein classes and explicit reciprocity laws*, Camb. J. Math. **5** (2017), no. 1, 1–122.

[KLZ20] ———, *Rankin-Eisenstein classes for modular forms*, Amer. J. Math. **142** (2020), no. 1, 79–138.

[Kol88] Victor Kolyvagin, *Finiteness of $E(\mathbf{Q})$ and Sha(E,$\mathbf{Q}$) for a subclass of Weil curves*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 3, 522–540, 670–671.

[Lam] Jack Lamplugh, *An Euler system for characters over an imaginary biquadratic field*, available at https://arxiv.org/abs/1611.05702.

[LLZ14] Antonio Lei, David Loeffler, and Sarah Zerbes, *Euler systems for Rankin-Selberg convolutions of modular forms*, Ann. of Math. (2) **180** (2014), no. 2, 653–771.

[LLZ15] ———, *Euler systems for modular forms over imaginary quadratic fields*, Compos. Math. **151** (2015), no. 9, 1585–1625.

[LV10] Matteo Longo and Stefano Vigni, *On the vanishing of Selmer groups for elliptic curves over ring class fields*, J. Number Theory **130** (2010), no. 1, 128–163.

[Miy89] Toshitsune Miyake, *Modular forms*, Springer-Verlag, Berlin, 1989, Translated from the Japanese by Yoshitaka Maeda.

[Mom81] Fumiyuki Momose, *On the ℓ-adic representations attached to modular forms*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 1, 89–109.

[MR04] Barry Mazur and Karl Rubin, *Kolyvagin systems*, Mem. Amer. Math. Soc. **168** (2004), no. 799, viii+96. MR 2031496 (2005b:11179)

[Nek00] Jan Nekovář, *p-adic Abel-Jacobi maps and p-adic heights*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), CRM Proc. Lecture Notes, vol. 24, Amer. Math. Soc., Providence, RI, 2000, pp. 367–379.

[Nek07] Jan Nekovář, *The Euler system method for CM points on Shimura curves*, L-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, pp. 471–547.

[Nek12] Jan Nekovář, *Level raising and anticyclotomic Selmer groups for Hilbert modular forms of weight two*, Canadian Journal of Mathematics **64** (2012), no. 3, 588–668.

[Rib85] Kenneth Ribet, *On ℓ-adic representations attached to modular forms. II*, Glasgow Math. J. **27** (1985), 185–194.

[Rub00] Karl Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000, Hermann Weyl Lectures. The Institute for Advanced Study.

[SU14] Christopher Skinner and Eric Urban, *The Iwasawa main conjectures for* GL_2, Invent. Math. **195** (2014), no. 1, 1–277.

[Tam21] Matteo Tamiozzo, *On the Bloch–Kato conjecture for Hilbert modular forms*, Mathematische Zeitschrift **299** (2021), no. 1, 427–458.

[Tat79] John Tate, *Number theoretic background*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26.

[Tia03] Ye Tian, *Euler systems of CM points on Shimura curves*, ProQuest LLC, Ann Arbor, MI, 2003, Thesis (Ph.D.)–Columbia University.

[Vat03] Vinayak Vatsal, *Special values of anticyclotomic L-functions*, Duke Math. J. **116** (2003), no. 2, 219–261.

[Wan15] Xin Wan, *The Iwasawa Main Conjecture for Hilbert modular forms*, Forum of Mathematics, Sigma **3** (2015), e18.

[Wan23] Haining Wang, *On the anticyclotomic Iwasawa main conjecture for Hilbert modular forms of parallel weights*, Annales mathématiques du Québec **47** (2023), no. 1, 195–248.

[Wil88] Andrew Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. **94** (1988), no. 3, 529–573.

[Wil95] ———, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551.

[Zha97] Shouwu Zhang, *Heights of Heegner cycles and derivatives of L-series*, Invent. Math. **130** (1997), no. 1, 99–152.

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